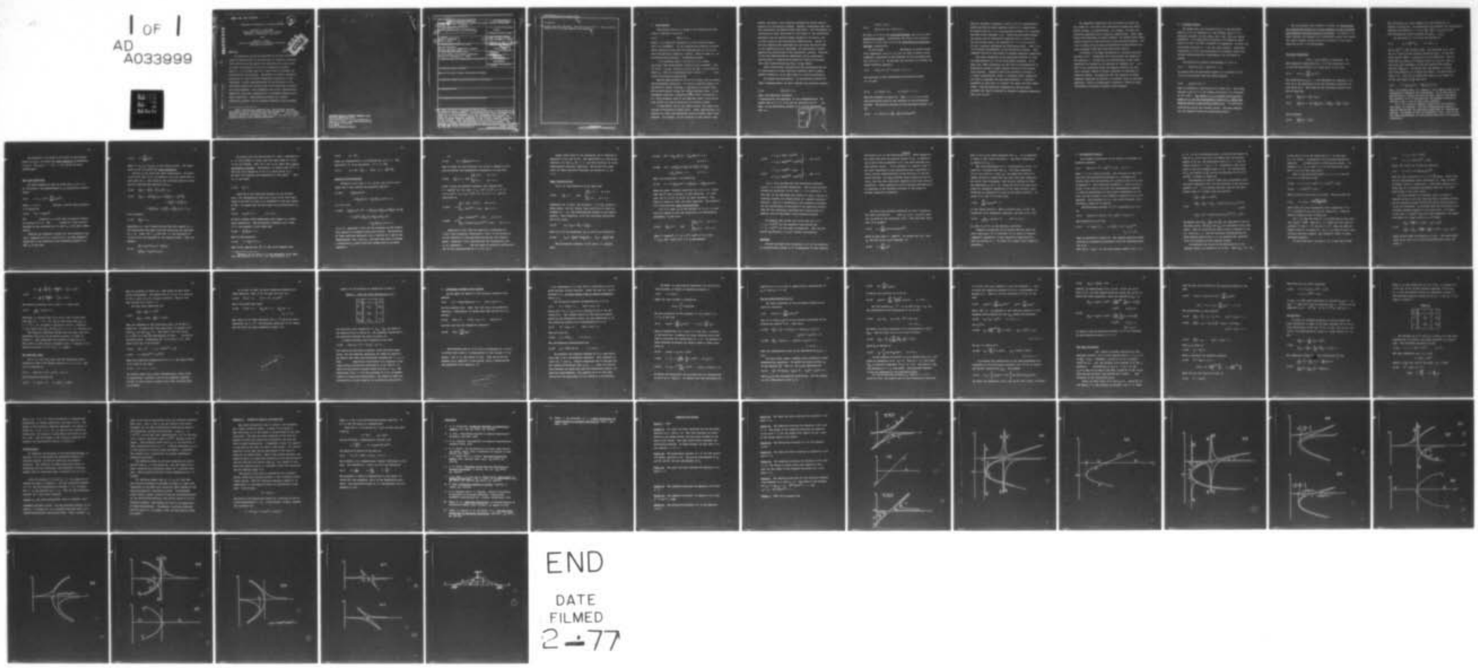
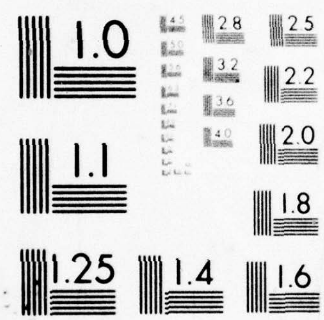
AD-A033 999   NEW YORK UNIV N Y COURANT INST OF MATHEMATICAL SCIENCES   F/G 12/1
SINGULAR PERTURBATIONS OF BIFURCATIONS,(U)
1976   B J MATKOWSKY, E L REISS   AF-AFOSR-2107-71

UNCLASSIFIED   AFOSR-TR-76-1307   NL

1 OF 1
AD
A033999



END
DATE
FILMED
2-77



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

Singular Perturbations of Bifurcations*

Bernard J. Matkowsky
Department of Mathematical Sciences
Rensselaer Polytechnic Institute

and

Edward L. Reiss
Courant Institute of Mathematical Sciences
New York University

Abstract

An asymptotic theory is presented to analyze perturbations of bifurcations of the solutions of nonlinear problems. The perturbations may result from imperfections, impurities, or other inhomogeneities in the corresponding physical problem. It is shown that for a wide class of problems the perturbations are singular. The method of matched asymptotic expansions is used to obtain asymptotic expansions of the solutions. Global representations of the solutions of the perturbed problem are obtained when the bifurcation solutions are known globally. This procedure also gives a quantitative method for analyzing singularities of nonlinear mappings and their unfoldings. Applications are given to a simple elasticity problem, and to nonlinear boundary value problems.

---

*This research was supported by: the National Science Foundation, grants MPS75-08328 and MCS75-07868 A01; the Office of Naval Research, contract N00014-76-C-0010; the Air Force Office of Scientific Research, grant 71-2107; and the U.S. Army Research Office, grant DAAG29-76-G-0253.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DDC
This technical report has been reviewed and is approved for public release IAW AFR 190-12 (7b).
Distribution is unlimited.
A. D. BLOSE
Technical Information Officer

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 18 AFOSR - 19 TR - 76 - 1307	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 SINGULAR PERTURBATIONS OF BIFURCATIONS,		5. TYPE OF REPORT & PERIOD COVERED Interim
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Bernard J. Matkowsky, Edward L. Reiss		8. CONTRACT OR GRANT NUMBER(s) 15 VAF AFOSR -2107-71, N00014-76-C-0010
9. PERFORMING ORGANIZATION NAME AND ADDRESS New York University, Courant Institute of Mathematical Sciences, New York, New York 10012		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 16 2304 17 A4
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332		12. REPORT DATE 11 1976
		13. NUMBER OF PAGES 56 12 58p.
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An asymptotic theory is presented to analyze perturbations of bifurcations of the solutions of nonlinear problems. The perturbations may result from imperfections, impurities or other inhomogeneities in the corresponding physical problem. It is shown that for a wide class of problems the perturbations are singular. The method of matched asymptotic expansions is used to obtain asymptotic expansions of the solutions. Global representations of the solutions of the perturbed problem are obtained when the bifurcation solutions are known globally. This procedure also gives a quantitative method for analyzing singularities of nonlinear

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 099 907

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20 Abstract

mappings and their unfoldings. Applications are given to a simple elasticity problem, and to nonlinear boundary value problems.

1. Introduction.

Bifurcation theory is a study of the branching of solutions of nonlinear equations,

$$f[y;\lambda] = 0 , \tag{1.1}$$

where f is a nonlinear operator, y is the solution vector, and λ is a parameter. It is of particular interest in bifurcation theory to study how the solutions $y(\lambda)$ of (1.1) and their multiplicity change as λ varies. Thus we refer to λ as the bifurcation parameter. A bifurcation point of a solution branch $y(\lambda)$ of (1.1) is a point $[\lambda_o, y(\lambda_o)]$ from which another solution $y_1(\lambda)$ branches. That is, $y(\lambda_o) = y_1(\lambda_o)$, and $y(\lambda) \neq y_1(\lambda)$ for all λ in an interval about λ_o. Thus at a bifurcation point there is a transition in solution multiplicity.

Typical applications of bifurcation theory are in elastic and hydrodynamic stability. In elastic stability theory, bifurcation is called buckling, λ represents an applied load, and the bifurcation points are called buckling loads. In hydrodynamic stability, bifurcations are called transitions, λ is a flow parameter such as the Reynolds number, and bifurcation points are called transition or critical points.

In experiments and in real applications, the sharp transitions of bifurcation rarely occur. Small imperfections, impurities or other inhomogeneities tend to distort these transitions. For example, in the buckling of thin elastic rods,

plates, and shells, the classical mathematical theory usually results in a bifurcation problem. However, experiments show that the transitions are smooth rather than sharp. The discrepancy is attributed to small deviations in the shapes of the experimental specimens from the perfect shapes assumed in the theory, and to other physical and experimental imperfections. Indeed, it has been demonstrated experimentally that when the magnitudes of the imperfections are decreased, the experimental results approach the predictions of the bifurcation problem. Moreover, it has been demonstrated by experiments and by approximate calculations on "imperfection-sensitive" elastic structures that small imperfections may have a large effect.

Small experimental imperfections and inhomogeneities are *of significance in other nonlinear problems, such as hydro-*dynamic stability, as we shall show in a future publication.

To analyze the perturbation of bifurcations produced by small inhomogeneities, we shall consider the nonlinear problem

$$F[y;\lambda,\delta] = 0 , \tag{1.2}$$

where the additional parameter δ characterizes the magnitude of the inhomogeneities. We assume that as $\delta \to 0$, (1.2) and its solutions $y(\lambda,\delta)$ reduce to the bifurcation problem (1.1) and its solutions, that is,

$$(1.3)\qquad \begin{aligned} y(\lambda,0) &= y(\lambda)\ , \\ F[y(\lambda,0);\lambda,0] &= f[y(\lambda);\lambda]\ . \end{aligned}$$

We refer to (1.2) as the perturbed problem, and (1.1) as the bifurcation problem. Furthermore, we refer to the solution branches of (1.2) and (1.1) as the perturbed and bifurcation branches, respectively.

In this paper, the method of matched asymptotic expansions, see e.g. [1,2,3], is used to obtain formal asymptotic expansions of the solutions of the perturbed problem (1.2) as $\delta \to 0$. To motivate our analysis, we consider the simple algebraic equation,

$$(1.4)\qquad F[y;\lambda,\delta] \equiv y^2 + (1-\lambda)y - \lambda\delta = 0\ .$$

The solutions of the corresponding bifurcation problem ($\delta = 0$) are,

$$(1.5)\qquad y = Y_0(\lambda) \equiv 0\ , \qquad y = Y_1(\lambda) \equiv \lambda - 1\ .$$

They are sketched in Figure 1b. Thus, $\lambda = 1$, $y = 0$ is the only bifurcation point of the solutions of the bifurcation problem. The solution branches of the perturbed problem (1.4) are

$$(1.6)\qquad y = y^{\pm}(\lambda,\delta) \equiv \frac{\lambda-1}{2} \pm \frac{1}{2}[(\lambda-1)^2+4\lambda\delta]^{1/2}\ .$$

They are sketched in Figures 1a and 1c for two representative small positive and small negative values of δ, respectively. We observe that as $\delta \to 0$, the perturbed branches (1.6) approach the bifurcation branches (1.5). Since the perturbed branches are smooth and the bifurcation branches have a "corner" at the bifurcation point, the approach as $\delta \to 0$ is non-uniform in any λ interval containing the bifurcation point. Thus (1.4) is a singular perturbation of the bifurcation problem. This suggests the use of the method of matched asymptotic expansions.

The outer expansions of the perturbed branches are regular perturbations in δ of the bifurcation branches. As we shall see, these expansions are not valid in neighborhoods of bifurcation points or other singular points of the bifurcation branches. (Singular points are defined in Section 2.) In these neighborhoods, inner, or boundary layer, expansions are used to represent the perturbed branches. A matching procedure is then used to "connect" the inner and outer expansions. Uniform asymptotic expansions of the perturbed branches can then be obtained by forming a composite expansion; see, e.g. [1,2,3].

The asymptotic expansions that we obtain are valid for any ranges of λ for which the bifurcation branches are known, either exactly, or approximately, for example, by other perturbation expansions or by numerical computations. Thus, when bifurcation branches are known globally, our method gives global representations of the solutions of the perturbed problem. Previous studies [4-8] of such perturbations of bifurcations are local analyses valid only near bifurcation points.

In Section 2, we apply the method to a general nonlinear equation (1.2). Relatively mild restrictions are imposed on the operator F. We show how the determination of the inner expansions is reduced to the analysis of algebraic equations. This is not surprising since the Lyapunov-Schmidt [9], and other theories, effect a similar reduction in classical bifurcation theory. We emphasize that the analysis of these algebraic equations is generic to a general class of nonlinear equations. This is illustrated in Section 4, where we apply the method to nonlinear boundary value problems.

2. A General Theory.

We assume that, the solutions $y(\lambda,\delta)$ of the perturbed problem (1.2) are elements of a real Hilbert space, F is a nonlinear operator on that space, and λ and δ are real valued parameters. For simplicity of notation, the possible dependence of F and y on other variables is suppressed. We shall use the bracket notation $\langle\cdot,\cdot\rangle$ for the inner product. Sufficient conditions on F for the application of our method, other than obvious differentiability requirements, will be stated as needed.

The bifurcation problem corresponding to (1.2) is

(2.1) $$F[y(\lambda,0);\lambda,0] = f[y(\lambda),\lambda] = 0 .$$

We assume that the bifurcation points of a solution $y(\lambda)$ of (2.1) are determined from the linear problem,

(2.2) $$f_y[y(\lambda);\lambda]\phi = 0$$

that is obtained by linearizing (2.1) about $y(\lambda)$. The linear operator $f_y[y(\lambda);\lambda]$ is the Frechet derivative of f evaluated at $y(\lambda)$ and λ. <u>The values $\lambda = \lambda_o$ for which (2.2) has solutions $\phi \not\equiv 0$, and the corresponding values $y(\lambda_o)$, define the singular points of the solution branch $y(\lambda)$</u>. In particular, bifurcation points are singular points. For many problems all the singular points are bifurcation points.

For the analysis that follows, we define the sub-branches of a bifurcation branch $y(\lambda)$ as the segments of $y(\lambda)$ connecting two successive singular points of the branch. In addition, we shall consider the segments of a bifurcation branch that connect the "last" singular points with the end points of the branch, which may be at ∞, as sub-branches.

The Outer Expansions.

Let $y = y_0(\lambda)$ denote a sub-branch. We seek asymptotic expansions as $\delta \to 0$ of the solutions $y(\lambda,\delta)$ of the perturbed problem (1.2) near $y_0(\lambda)$ in the form,

$$(2.3) \qquad y(\lambda,\delta) = \sum_{j=0}^{\infty} y_j(\lambda)\delta^j ,$$

The coefficients $y_1, y_2, \ldots,$ are determined by inserting (2.3) into (1.2) and equating to zero the resulting coefficient of each power of δ. This leads to the following recursive system of equations

$$(2.4) \qquad F_y^o y_1(\lambda) = -F_\delta^o \equiv R_1(\lambda)$$

$$(2.5) \qquad F_y^o y_2(\lambda) = -\tfrac{1}{2}\,[(F_{yy}^o y_1)y_1 + 2F_{y\delta}^o y_1 + F_{\delta\delta}^o] \equiv R_2(\lambda)$$

and in general

$$(2.6) \qquad F_y^o y_j(\lambda) = R_j(\lambda)$$

The derivatives of F with respect to y are functional, or Fréchet, derivatives. The superscript zero denotes the corresponding operator evaluated at $\delta = 0$, e.g. $F_y^o = F_y^o(\lambda) \equiv F_y[y_0(\lambda);\lambda,0]$. The inhomogeneous terms R_j depend on $y_0, y_1, \ldots, y_{j-1}$.

The solutions of (2.4)-(2.6) are written as

$$(2.7) \qquad y_j = (F_y^o)^{-1} R_j \; , \qquad j = 1,2,\ldots \; ,$$

where $(F_y^o)^{-1}$ is the inverse of F_y^o. The solutions (2.7) exist for all λ that are not singular points of $y_0(\lambda)$. At the singular points, $F_y^o(\lambda) = f_y[y_0(\lambda);\lambda]$ is not invertible. The coefficients y_j exist at a singular point λ_o of $y_0(\lambda)$, only if the $R_j(\lambda_o)$ are orthogonal to the null space of the adjoint operator $(F_y^o(\lambda_o))^*$. We assume that the operator F is such that this condition is violated *for some* j. In particular we take j = 1. That is, we assume that there is an element ϕ^* of the null space of F_y^{o*} such that

$$(2.8) \qquad I \equiv \langle R_1(\lambda_o), \phi^* \rangle = -\langle F_\delta^o(\lambda_o), \phi^* \rangle \neq 0 \; .$$

Then, as we shall show, c.f. Figure 1, (2.8) implies that the inhomogeneities "destroy" the bifurcations. It is this type of perturbation of bifurcation that we shall study in this paper. The condition (2.8) is satisfied for the problems analyzed in Sections 3 and 4, and for many other problems.

If I = 0, a similar analysis, which we do not present in this paper, is required. Then the inhomogeneities need not destroy bifurcation; the perturbed problem may be a bifurcation problem. An example of this is discussed in [10]. See [11,12] for other perturbations of bifurcations that result in secondary bifurcation.

The expansion (2.3), which is not valid at the singular points of $y_0(\lambda)$, is called the outer expansion corresponding to $y_0(\lambda)$. The $y_j(\lambda)$, $j = 1,2,\ldots,$ are called the outer coefficients.

The Inner Expansions.

We obtain expansions that are valid near λ_o as $\delta \to 0$, by "stretching" the neighborhoods of λ_o through the transformation

$$(2.9) \qquad \lambda = \lambda_o + \xi\mu^a + \sum_{i=2}^{\infty} \xi_i (\mu^a)^i ,$$

where $a > 0$ is a constant, and the small parameter μ is defined by

$$(2.10) \qquad \delta(\mu) = (\mathrm{sgn}\,\delta)\mu^b .$$

The constant b in (2.10) must be positive because we require $\delta(0) = 0$. The values of a and b are determined by the nonlinearity of F near λ_o, as we shall demonstrate.

Denoting the dependent variable for the analysis of the inner expansion by $z(\mu) = y(\lambda(\mu),\delta(\mu))$, we seek asymptotic expansions of the solutions of the perturbed problem (1.2) near λ_o in the form

$$z = \sum_{j=0}^{\infty} z_j \mu^j , \tag{2.11}$$

where $\lambda = \lambda_o$, $z_0 = y_0(\lambda_o)$ is the singular point. The expansions (2.11) are called the <u>inner expansions</u>,

and the z_j are called the inner coefficients. We insert (2.9)-(2.11) into (1.2) and equate to zero the coefficient of each power of μ. This leads to the following system of equations to successively determine the z_j:

$$F_z^o z_1 = -F_\lambda^o \lambda_\mu^o - F_\delta^o(\lambda_o)\delta_\mu^o \equiv \rho_1 , \tag{2.12}$$

$$\begin{aligned} F_z^o z_2 = -\tfrac{1}{2}\{(F_{zz}^o z_1)z_1 &+ 2\lambda_\mu^o F_{z\lambda}^o z_1 + (\lambda_\mu^o)^2 F_{\lambda\lambda}^o \\ &+ \lambda_{\mu\mu}^o F_\lambda^o + \delta_{\mu\mu}^o F_\delta^o(\lambda_o) \\ &+ \delta_\mu^o[2F_{z\delta}^o z_1 + 2\lambda_\mu^o F_{\lambda\delta}^o + \delta_\mu^o F_{\delta\delta}^o(\lambda_o)]\} \equiv \rho_2 , \end{aligned} \tag{2.13}$$

and in general,

$$F_z^o z_j = \rho_j . \tag{2.14}$$

Subscripts on λ and δ denote derivatives with respect to μ. The superscript zero means that the quantities are evaluated at $\mu = 0$. Since $\lambda(0) = \lambda_o$ and $\delta(0) = 0$, the operators in (2.12)-(2.14) are evaluated at the singular point. Thus, for example,

$$\begin{aligned} F_z^o &= F_z[z(0);\lambda_o,0] = F_y^o(\lambda_o) , \\ F_\delta^o(\lambda_o) &= F_\delta[z(0);\lambda_o,0] . \end{aligned} \tag{2.15}$$

We require that the derivatives of λ and δ, evaluated at $\mu = 0$, are bounded to insure that the right sides of (2.12)-(2.14) are bounded. Then (2.9) and (2.10) imply that <u>a and b are positive integers</u>. Furthermore, we observe that if $b = 1$, then the inner expansion (2.11) is a power series in δ. We shall not consider such expansions in this paper.* Thus $b \geq 2$, and hence,

$$(2.16) \qquad \delta_\mu^o = 0 .$$

Since F_z^o is the linearized operator at the singular point, the inhomogeneous equations (2.12)-(2.14) have solutions if and only if each ρ_j is orthogonal to the null space of F_z^{o*}. We assume that the null space is one-dimensional, so that

$$(2.17) \qquad \langle \rho_j , \phi^* \rangle = 0 , \qquad j = 1,2,\ldots .$$

We shall consider multi-dimensional null spaces in a subsequent publication. The solvability condition (2.17) with $j = 1$, and equation (2.16) imply that

$$(2.18) \qquad \lambda_\mu^o \langle F_\lambda^o , \phi^* \rangle = 0 .$$

Thus if the projection

$$(2.19) \qquad \Lambda \equiv \langle F_\lambda^o , \phi^* \rangle \neq 0 ,$$

then (2.18) implies that $\lambda_\mu^o = 0$, and (2.12) implies that

---

*Equation (2.17) with $j = 1$, and assumption (2.8) imply that the expansion (2.9) for λ is a power series in δ.

$$z_1 = A\phi \ , \tag{2.20}$$

where the eigenfunction ϕ is normalized by $\langle\phi,\phi\rangle = 1$. The amplitude A is to be determined. If $\Lambda = 0$, then

$$z_1 = A\phi \dotplus \lambda_\mu^o Z_1 \ , \quad \text{where} \quad Z_1 \equiv -(F_z^o)^{-1}F_\lambda^o \ . \tag{2.21}$$

Quadratic Nonlinearities.

Condition (2.17) with j = 2, (2.16), and (2.19)-(2.21) imply that A must satisfy the quadratic equation

$$\begin{aligned} &\langle (F_{zz}^o\phi)\phi,\phi^*\rangle A^2 \\ &\qquad + \{\lambda_{\mu\mu}^o\langle F_\lambda^o,\phi^*\rangle + \delta_{\mu\mu}^o\langle F_\delta^o(\lambda_o),\phi^*\rangle\} = 0 \ , \end{aligned} \tag{2.22a}$$

if $\Lambda \neq 0$, and

$$\begin{aligned} &\langle (F_{zz}^o\phi)\phi,\phi^*\rangle A^2 + \langle[(F_{zz}^o\phi)Z_1+(F_{zz}^oZ_1)\phi+2F_{z\lambda}^o\phi],\phi^*\rangle\lambda_\mu^o A \\ &\quad + \{(\lambda_\mu^o)^2\langle[(F_{zz}^oZ_1)Z_1+2F_{z\lambda}^oZ_1+F_{\lambda\lambda}^o],\phi^*\rangle \\ &\qquad\qquad + \delta_{\mu\mu}^o\langle F_\delta^o(\lambda_o),\phi^*\rangle\} = 0 \ , \end{aligned} \tag{2.22b}$$

if $\Lambda = 0$. Equations (2.22) are the analogues of the bifurcation equation of standard bifurcation theory. The solutions of these equations determine A as a function of λ and of the inhomogeneity, for λ near λ_o. For each real root, we obtain a coefficient z_1, which gives the leading term in an inner expansion.

If

$$(2.23) \qquad Q \equiv \langle (F^o_{zz}\phi)\phi, \phi^* \rangle \neq 0 ,$$

then in order for the solutions A of (2.22) to depend on both the bifurcation and inhomogeneity parameters, we must have

$$(2.24) \qquad \delta^o_{\mu\mu} \neq 0 , \quad \text{and} \quad \begin{cases} \lambda^o_{\mu\mu} \neq 0 , & \text{if } \Lambda \neq 0 , \\ \lambda^o_{\mu} \neq 0 , & \text{if } \Lambda = 0 . \end{cases}$$

Since a and b are positive integers, this implies that $\delta = (\mathrm{sgn}\delta)\mu^2$ (b = 2), and $\lambda = \lambda_o + \xi\mu^2 + 0(\mu^3)$ (a = 2) if $\Lambda \neq 0$, and $\lambda = \lambda_o + \xi\mu + 0(\mu^2)$ (a = 1) if $\Lambda = 0$. This is equivalent to

$$(2.25a) \qquad \lambda = \begin{cases} \lambda_o + \xi|\delta| + 0(\delta^2) , & \text{for } \Lambda \neq 0 , \\ \lambda_o + \xi|\delta|^{1/2} + 0(\delta), & \text{for } \Lambda = 0 ; \end{cases}$$

$$(2.25b) \qquad z = \begin{cases} z_0 + A(\xi)\phi|\delta|^{1/2} + 0(\delta) , & \text{for } \Lambda \neq 0 , \\ z_0 + (A(\xi)\phi + \xi z_1)|\delta|^{1/2} + 0(\delta) , & \text{for } \Lambda = 0 . \end{cases}$$

Equations (2.25), with the amplitude A determined by (2.22), give parametric descriptions, with ξ as the parameter, of the solutions of the perturbed problem near the singular point. Condition (2.23) characterizes the nonlinearity near λ_o as quadratic. The two types of quadratic nonlinearity are in turn characterized by $\Lambda = 0$ or $\Lambda \neq 0$.

6

Higher order terms in the expansions can be obtained by analyzing (2.13) and (2.14). The amplitudes A_j of the eigenfunction ϕ in each z_j for $j > 1$, are then solutions of linear inhomogeneous algebraic equations. The solvability conditions for these equations determine the parameters ξ_j for $j \geq 2$.

Cubic Nonlinearities.

If $Q = 0$, then Equations (2.22) imply that

$$(2.26) \qquad \delta^{o}_{\mu\mu} = 0 \,, \quad \text{and} \quad \begin{cases} \lambda^{o}_{\mu\mu} = 0 \,, & \text{if } \Lambda \neq 0 \\ \lambda^{o}_{\mu} = 0 \,, & \text{if } \Lambda = 0 \,. \end{cases}$$

Otherwise, for (2.22a), the variable $\xi = 1/2\ \lambda^{o}_{\mu\mu}$ would be a fixed number, and for (2.22b) there would be no A which is bounded as $\lambda \to \lambda_o$, and simultaneously depends on the inhomogeneity. Hence Equations (2.22) are satisfied identically and (2.13) gives

$$(2.27) \qquad z_2 = A_2\phi + A^2 Z_2 + \lambda^{o}_{\mu\mu}\hat{Z}_2 \,,$$

where A_2 is to be determined, and Z_2 and $\hat{Z}_2$ are defined by

$$(2.28) \qquad Z_2 \equiv -\tfrac{1}{2}(F^{o}_{z})^{-1}[(F^{o}_{zz}\phi)\phi] \,, \quad \hat{Z}_2 \equiv -\tfrac{1}{2}(F^{o}_{z})^{-1}F^{o}_{\lambda} \,.$$

The solvability condition (2.17) with $j = 3$ implies that,

$$\text{(2.29a)} \qquad CA^3 + \{\lambda^o_{\mu\mu\mu}\langle F^o_\lambda,\phi^*\rangle + \delta^o_{\mu\mu\mu}\langle F^o_\delta,\phi^*\rangle\} = 0 \ , \qquad \text{if } \Lambda \neq 0 \ ,$$

$$\text{(2.29b)} \qquad CA^3 + \langle[2F^o_{zz}\hat{Z}_2\phi + 4(F^o_{zz}\phi)\hat{Z}_2 + 3F^o_{z\lambda}\phi],\phi^*\rangle\lambda^o_{\mu\mu}A + \delta^o_{\mu\mu\mu}\langle F^o_\delta,\phi^*\rangle = 0 \ , \qquad \text{if } \Lambda = 0 \ ,$$

where the coefficient C is defined by,

$$\text{(2.30)} \qquad C \equiv \langle\{((F^o_{zzz}\phi)\phi)\phi + 4(F^o_{zz}\phi)Z_2 + 2(F^o_{zz}Z_2)\phi\},\phi^*\rangle \ .$$

These are cubic algebraic equations for A if $C \neq 0$. Notice that the A^2 term is missing in both equations. For each real root of (2.29) we obtain an inner expansion. Thus there is always at least one inner expansion. The condition $C \neq 0$ characterizes <u>cubic nonlinearities</u>.

If $C \neq 0$, then, as in the analysis of (2.22), in order for A to depend on both the bifurcation and inhomogeneity parameters, we must have

$$\text{(2.31)} \qquad \delta^o_{\mu\mu\mu} \neq 0 \ , \quad \text{and} \quad \begin{cases} \lambda^o_{\mu\mu\mu} \neq 0 \ , & \text{if } \Lambda \neq 0 \ , \\ \lambda^o_{\mu\mu} \neq 0 \ , & \text{if } \Lambda = 0 \ . \end{cases}$$

Then $\delta = (\operatorname{sgn}\delta)\mu^3$, $\lambda = \lambda_o + \xi\mu^3 + 0(\mu^4)$ if $\Lambda \neq 0$ and $\lambda = \lambda_o + \xi\mu^2 + 0(\mu^3)$ if $\Lambda = 0$, or equivalently

$$\left.\begin{aligned} \lambda &= \lambda_o + \xi|\delta| + O(\delta^{4/3}) \\ z &= z_0 + A\phi|\delta|^{1/3} + O(\delta^{2/3}) \end{aligned}\right\}, \quad \text{for } \Lambda \neq 0\,, \tag{2.32}$$

$$\left.\begin{aligned} \lambda &= \lambda_o + \xi|\delta|^{2/3} + O(\delta) \\ z &= z_0 + A\phi|\delta|^{1/3} + O(\delta^{2/3}) \end{aligned}\right\}, \quad \text{for } \Lambda = 0\,. \tag{2.33}$$

If $C = 0$, we can show that the solvability condition $\langle\rho_3,\phi^*\rangle = 0$ is satisfied identically. Then we must consider solvability conditions (2.17) with $j > 3$ to evaluate A. In this way, we can determine the values of a and b, and we can formally classify the singularities of a general nonlinear equation. This is a by-product of our asymptotic procedure for the construction of solutions of the perturbed problem. Thom [8] has rigorously classified the singularities of a special class of equations, using different methods.

We observe from (2.25a) and (2.33) that for $\Lambda = 0$, $\lambda - \lambda_o = O(\delta^{1/2})$ for the quadratic nonlinearity and $\lambda - \lambda_o = O(\delta^{2/3})$ for the cubic nonlinearity. Thus the quadratic nonlinearity is a more "severe" perturbation.

Matching.

We have obtained outer expansions (2.3) of the solutions of the perturbed problem (1.2) corresponding to each solution

sub-branch $y_0(\lambda)$ of the bifurcation problem (2.2). These expansions are valid away from the singular points of y_0. In addition, one or more inner expansions (2.11) have been obtained near each singular point. It is necessary to "connect" each of the outer expansions to the appropriate inner expansions, to obtain uniform representations of the solutions of the perturbed problem. For many problems these connections are obvious, once the inner and outer expansions have been obtained explicitly. These connections can be obtained systematically by appealing to the matching procedure of the method of matched asymptotic expansions.

We derive the matching conditions by first considering the outer coefficients near λ_o, $y_j(\lambda) = y_j(\lambda(\mu))$, where $\lambda(\mu)$ is given by the stretching (2.9). Then the inner expansion (2.3) becomes

$$(2.34) \qquad y = \sum_{j=0}^{\infty} y_j(\lambda(\mu))(\mathrm{sgn}\delta)^j \mu^{jb} ,$$

where we have used $\delta = (\mathrm{sgn}\delta)\mu^b$. We assume that for λ near λ_o, the sum (2.34) can be expanded as

$$(2.35) \qquad y = \sum_{k=0}^{\infty} \eta_k \mu^k .$$

Thus (2.35) is the outer expansion near λ_o. It is expressed in terms of the "inner variables". The outer coefficients η_k depend on $\xi, \xi_2, \ldots$.

The outer expansions (2.3), or equivalently (2.35), are valid for λ bounded away from λ_o. The inner expansions (2.11) are valid for λ near λ_o. We assume that there is a common interval in which both the inner and outer expansions are valid; and furthermore, this interval shrinks in width and approaches λ_o as $\delta \to 0$. Since the inner and outer expansions are asymptotic expansions of the same function, their difference is asymptotic to zero in the common interval. By using (2.11) and (2.35) this gives

$$(2.36) \qquad \sum_{k=0}^{\infty} (\eta_k - z_k)\mu^k \sim 0$$

in the common interval. Then it follows from, (2.36), the definition of an asymptotic expansion, and from (2.9), that

$$(2.37) \qquad \lim_{|\xi| \to \infty} [\eta_k - z_k] = 0 , \qquad \text{for } k = 1,2,\ldots .$$

We refer to (2.37) as the matching conditions.

Composite expansions can be formed from the inner and outer expansions, as is customary in the method of matched asymptotic expansions, to give asymptotic expansions in δ that are uniform in λ. We shall not present these composite expansions.

3. An Algebraic Problem.

As a simple illustration of our method, we consider the algebraic equation

$$(3.1)\qquad F[y;\lambda,\delta] = y^3 + 2\beta y^2 + (1-\lambda)y - \lambda\delta = 0 \ ,$$

where $\beta > 0$ is a given constant. The solutions of this equation give the equilibrium positions of a one-degree-of-freedom model for the buckling of an elastic arch, as we show in Appendix I. The bifurcation parameter λ, which is proportional to the load on the arch, the inhomogeneity parameter δ, which is proportional to the initial imperfection in the arch, and the constant β are defined in Equation (A.2) of the Appendix. The solutions of (3.1) are easily obtained, thus providing a check of our method.

The bifurcation problem ($\delta = 0$) corresponding to (2.1) is

$$(3.2)\qquad F[y;\lambda,0] = f[y,\lambda] = y^3 + 2\beta y^2 + (1-\lambda)y = 0 \ .$$

The solutions of (3.2) are

$$(3.3)\qquad y = Y_0(\lambda) \equiv 0 \ , \quad y = Y_1^{\pm}(\lambda) = -\beta \pm (\lambda-\lambda_\ell)^{1/2} \ ,$$

$$\lambda_\ell \equiv 1 - \beta^2 \ .$$

They are sketched in Figure 2b. The singular points of these bifurcation branches are determined from the linearized problem (2.2).

Thus for $y = Y_0(\lambda) \equiv 0$, the only singular point is at $\lambda = 1$,

$y = 0$. It is a bifurcation point, as we see from Figure 2b. From (2.2), (3.2) and (3.3), we deduce that the singular points of Y_1^+ are, the bifurcation point at $\lambda = 1$, and the "limit" point at $\lambda = \lambda_\ell$, $y = -\beta$. Furthermore, this limit point is the only singular point of Y_1^-. The sub-branches of Y_o are denoted by $Y_o^\pm$, and the sub-branches of Y_1^+ by Y_1^{++} and Y_1^{+-}, as shown in Figure 2b.

Corresponding to each of these five sub-branches there are outer expansions (2.3) of the solutions of the perturbed problem (3.1). In an obvious notation, and for the appropriate range of λ values, they are given by

$$y^\pm_{(0)} = \frac{\lambda}{1-\lambda}\,\delta + 0(\delta^2) ,$$

$$(3.4)\qquad y^{++}_{(1)} = y^{+-}_{(1)} = Y^+_{(1)} + (\lambda/2)\{\lambda-\lambda_\ell-\beta(\lambda-\lambda_\ell)^{1/2}\}^{-1}\delta + 0(\delta^2) ,$$

$$y^-_{(1)} = Y_1^- + (\lambda/2)\{\lambda-\lambda_\ell+\beta(\lambda-\lambda_\ell)^{1/2}\}^{-1}\delta + 0(\delta^2) .$$

We observe that $y^\pm_{(0)}$, $y^{++}_{(1)}$ and $y^{+-}_{(1)}$ are unbounded at the bifurcation point, and $y^{+-}_{(1)}$ and $y^-_{(1)}$ are unbounded at the limit point. They are shown by the solid curves in Figures 2a and 2c. Since $F_\delta = -\lambda$, and the normalized eigenfunctions for both singular points are $\phi = 1$, we conclude that condition (2.8) is satisfied at both singular points.

To determine the nature of the nonlinearity at the singular points, we evaluate Q in (2.23). Since $F_{zz} = 6z + 4\beta$,

we see that $Q \neq 0$ at the bifurcation ($z = 0$) and limit ($z = -\beta$) points. Consequently, the perturbed problem has quadratic nonlinearities at both singular points. Furthermore, $F_\lambda = -y$, so that $\Lambda = 0$ at the bifurcation point, and $\Lambda = \beta \neq 0$ at the limit point.

Thus at the bifurcation point $\lambda_o = 1$, $y = 0$, we have $a = 1$, $b = 2$ and from (2.25),

$$\text{(3.5a)} \qquad \lambda = 1 + \xi\mu + 0(\mu^2) = 1 + \xi|\delta|^{1/2} + 0(\delta) ,$$

$$\text{(3.5b)} \qquad z = A|\delta|^{1/2} + 0(\delta) ,$$

where A are the real roots of (2.22b). They are given by,

$$\text{(3.6)} \qquad A = A^{\pm}(\xi) \equiv \frac{\xi \pm \sqrt{\xi^2 + 8\beta(\operatorname{sgn}\delta)}}{4\beta} .$$

Hence there are two corresponding inner expansions $z^{\pm}$ for $\delta > 0$, because $A^{\pm}$ are real for all values of ξ. They are shown by dashed curves in Figure 2a.

If $\delta < 0$, then for real $A^{\pm}$, ξ is restricted to the intervals $|\xi| \geq \xi^o \equiv 2\sqrt{2\beta}$. Thus for $\delta < 0$, there are four inner expansions, $A^{\pm}$ for $\xi > \xi^o$, and for $\xi < \xi^o$. They are shown by dashed curves $z^{\pm}$ in Figure 2c. At $\xi = \xi_o$, both z^+ and z^- join and have vertical tangents, thus forming a "nose" of these perturbed branches.

At the limit point, we have $a = b = 2$ and from (2.25b)

$$\text{(3.7)}\quad \begin{aligned} \lambda &= \lambda_\ell + \xi|\delta| + O(\delta^2) \\ z &= -\beta + A(\xi)|\delta|^{1/2} + O(\delta) , \end{aligned}$$

where, the roots A of (2.22a) are given by,

$$\text{(3.8)}\qquad A = A^{\pm} = \pm[\xi-(\operatorname{sgn}\delta)\lambda_\ell/\beta]^{1/2} .$$

They are real if and only if $\xi \geq \xi^1 \equiv \frac{\lambda_\ell}{\beta}(\operatorname{sgn}\delta)$. Hence there are two corresponding inner expansions, $z_\ell^{\pm}$. They are shown by dashed curves in Figures 2a,c. From these figures, we observe that as δ increases (decreases) from zero, the limit point moves to the right (left).

For this problem it is obvious, and it is shown in Figures 2a and 2c, how the inner and outer expansions connect. We will therefore merely verify that the matching conditions (2.32) are satisfied. We consider only the bifurcation point. A similar analysis applies to the limit point. We insert (3.5a) and $\delta = (\operatorname{sgn}\delta)\mu^2$ into the outer expansions, and expand the result in a power series in μ, to obtain

$$\text{(3.9)}\quad \begin{aligned} y^{\pm}_{(0)} &= \frac{\operatorname{sgn}\delta}{\xi}\mu + O(\mu^2) = \eta_1\mu + O(\mu^2) \\ y^{++}_{(1)} = y^{+-}_{(1)} &= \left(\frac{\operatorname{sgn}\delta}{\xi} + \frac{\xi}{2\beta}\right)\mu + O(\mu^2) = \eta_1\mu + O(\mu^2) , \end{aligned}$$

where we have used the notation of (2.35). The inner expansions (3.5), and (3.6) at the bifurcation point give, for large ξ,

$$z^+ = \frac{\xi}{4\beta} + \frac{|\xi|}{4\beta}\left[1 + \frac{4\beta \operatorname{sgn}\delta}{\xi^2} + \cdots\right]\mu + O(\mu^2)$$

(3.10)

$$z^- = \frac{\xi}{4\beta}\left[-\frac{4\beta \operatorname{sgn}\delta}{\xi^2} + \cdots\right]\mu + O(\mu^2) \;.$$

The matching condition (2.37) with k = 1 states that

(3.11) $$\lim_{|\xi|\to\infty} (\eta_1 - z_1) = 0 \;.$$

Therefore, it follows from (3.9)-(3.11) that z^+ must match with $y^{++}_{(1)}$ as $\xi \to \infty$ ($\lambda \to \infty$), and it must match with $y^-_{(0)}$ as $\xi \to -\infty$ ($\lambda \to -\infty$), as shown in Figures 2a and 2c. Similarly, z^- must match with $y^+_{(0)}$ and $y^{+-}_{(1)}$ as $\xi \to \infty$, $\xi \to -\infty$, respectively.

The composite expansions formed from the inner and outer expansions then give the perturbed branches, as shown in Figures 3. This summarizes the analysis of Equation (3.1). The curves in these figures represent cuts δ = constant in the solution surfaces of (3.1) in y, λ, δ space.

The Canonical Cubic.

For $\beta = 0$, the limit point and the bifurcation point coalesce to form a new singular point at $\lambda = 1$, $y = 0$. Then (3.1) is reduced to,

(3.12) $$F[y;\lambda,\delta] = y^3 + (1-\lambda)y - \lambda\delta = 0 \;.$$

The bifurcation branches are

(3.13) $$y = Y_0(\lambda) \equiv 0 \;, \quad y = Y_1^{\pm}(\lambda) = \pm\sqrt{\lambda - 1} \;.$$

They are sketched in Figure 4a. Thus there are four bifurcation sub-branches. We observe that if $y(\lambda,\delta)$ is a solution of (4.1), then $-y(\lambda,-\delta)$ is also a solution. Thus we need only analyze (4.1) with $\delta > 0$.

The four outer expansions are

$$(3.14)\qquad \begin{aligned} y^{\pm}_{(0)} &= \frac{\lambda}{1-\lambda}\,\delta + O(\delta^2) \\ y^{\pm}_{(1)} &= \pm\sqrt{\lambda-1} - \frac{\lambda/2}{1-\lambda}\,\delta + O(\delta^2)\ . \end{aligned}$$

They are unbounded at the bifurcation point, as we show in Figure 4b. To obtain the inner expansions, we observe first that $F_{zz} = 6z$ and $F_{zzz} = 6$. Thus from (2.23) and (2.30), $Q = 0$ and $C \neq 0$ and the nonlinearity is cubic at the bifurcation point. Furthermore $F^{0}_{\lambda} = 0$, so that $\Lambda = 0$. Thus from (2.31) and (2.33) we have,

$$(3.15)\qquad \delta(\mu) = \mu^3\ , \quad \lambda = 1 + \xi|\delta|^{2/3} + O(\delta)$$

$$(3.16)\qquad z = A(\xi)|\delta|^{1/3} + O(\delta^{2/3})$$

where the normalized eigenfunction is $\phi = 1$, and from (2.29b), A is a root of the cubic

$$(3.17)\qquad A^3 - \xi A - 1 = 0\ .$$

Our method yields only a minor simplification, since (3.12) is essentially a canonical form of the algebraic equation (2.29b) to which general problems with cubic nonlinearities are reduced.

It is easy to show, by using elementary properties of cubic equations, that (3.17) has only one real root

$$(3.18) \qquad A^{+}(\xi) > 0 \,, \qquad \text{if } \xi < \xi^{o} \equiv 3/(2)^{2/3} \,,$$

and it has three real roots

$$(3.19) \qquad A^{+}(\xi) > 0 \,, \quad A^{-}_{(1)}(\xi) < 0 \,, \quad A^{-}_{(2)}(\xi) < 0 \,, \qquad \text{if } \xi > \xi^{o} \,.$$

Thus there is one inner expansion for $\xi < \xi^{o}$ and three inner expansions for $\xi > \xi^{o}$. The matching conditions (2.37) decide how the inner and outer expansions connect. The

Go to next page

results of the matching are summarized in Table I.

Table I. Inner and Outer Matching as $\delta \to 0$.

$\xi \to$	∞	$-\infty$
A^{+}	$Y^{+}_{(1)}$	$Y^{-}_{(0)}$
$A^{-}_{(1)}$	$Y^{+}_{(0)}$	—
$A^{-}_{(2)}$	$Y^{-}_{(1)}$	—

The resulting inner expansions z^{+}, $z^{-}_{(1)}$, $z^{-}_{(2)}$ are shown by the dashed curves in Figure 4b. The composite expansions of the perturbed branches are shown in Figure 4c.

A similar analysis can be applied to the cubic

$$F[y;\lambda,\delta] = y^3 - (1-\lambda)y - \lambda\delta = 0 \ . \tag{3.20}$$

The resulting bifurcation branches, inner and outer expansions, and the composite expansions are shown in Figures 5. We observe that there are bifurcation branches for $\lambda < \lambda_o = 1$. This subcritical bifurcation is in contrast to the supercritical bifurcation exhibited by (3.12) with $\delta = 0$, where the bifurcation branches exist only for $\lambda > \lambda_o$. The bifurcation from $\lambda = 1$ of the solutions of (3.2) is another example of subcritical bifurcation. The effect of small inhomogeneities is most dramatic for subcritical bifurcation.

4. A Nonlinear Boundary Value Problem.

We now apply the method to the nonlinear boundary value problem,

$$y'' + \lambda[G(y)+\delta g(x,y)] = 0 , \quad y(0) = y(\pi) = 0 , \tag{4.1}$$

for the function $y(x)$. Here, $G(y)$ and $g(x,y)$ are prescribed functions. Furthermore, we assume that they satisfy the conditions

$$G(0) = 0 , \quad G'(0) \equiv a_1 > 0 , \quad g(x,0) \not\equiv 0 , \tag{4.2a}$$

and that $G(y)$ has the asymptotic expansion,

$$G(y) = \sum_{j=1}^{\infty} a_j y^j . \tag{4.2b}$$

The bracketed term in (4.1) can be interpreted as a source, or growth term, where λ is proportional to the strength of the source. For $\delta = 0$, the source is $G(y)$. Then δg can be considered as an "impurity" in the source, where δ characterizes the magnitude of the impurity. If

Go to next page

g is independent of y, then $\delta g(x)$ is equivalent to an applied external forcing function. Since the sign of δ can be absorbed in g, we shall assume, with no loss of generality, that $\delta \geq 0$.

The bifurcation problem corresponding to (4.1) is

$$(4.3) \qquad y'' + \lambda G(y) = 0 , \qquad y(0) = y(\pi) = 0 .$$

Since $G(0) = 0$, $y = Y_0 \equiv 0$ is a solution of (4.3) for all values of λ. The singular points of this solution branch, which are all bifurcation points, are the eigenvalues λ_n of the linearized problem (2.2) corresponding to (4.3),

$$(4.4) \qquad \phi'' + \lambda a_1 \phi = 0 , \qquad \phi(0) = \phi(\pi) = 0 .$$

They are given by

$$(4.5a) \qquad \lambda_n = n^2/a_1 , \qquad n = 1,2,\ldots .$$

The corresponding eigenfunctions are

$$(4.5b) \qquad \phi_n = \sqrt{2/\pi} \sin nx , \qquad n = 1,2,\ldots .$$

We determine the solution branches of (4.3) that bifurcate from λ_1 by a perturbation expansion. This expansion is valid for λ near λ_1. Similar expansions can be obtained for the branches bifurcating from $\lambda_2, \lambda_3, \ldots$. Thus the bifurcation branches are known only near the bifurcation points, and then only approximately. The analysis of (4.1) will thus demonstrate the application of our method to such problems.

To obtain the perturbation expansions for the bifurcation branches, we define an amplitude parameter ε,

(4.6) $\varepsilon \equiv \langle y,\phi_1\rangle$,

where the inner product is defined by,

$$\langle f,g\rangle \equiv \int_0^\pi f(x)g(x)dx \ .$$

We seek expansions of the solutions of (4.3) near $y = 0$, $\lambda = \lambda_1$ in the form

(4.7) $y(x;\varepsilon) = \sum_{j=1}^{\infty} y_j(x)\varepsilon^j \ , \qquad \lambda = \lambda_1 + \sum_{1}^{\infty} \Lambda_j\varepsilon^j \ .$

Then by substituting (4.7) into (4.3) and (4.6), we obtain, in the usual way, a sequence of linear boundary value problems to determine the coefficients in (4.7). An analysis of these problems determines the leading terms in these expansions as

(4.8a) $y(x;\varepsilon) = \varepsilon\phi_1(x) + O(\varepsilon^2)$

(4.8b) $\lambda = \lambda_1 - \varepsilon \frac{8\sqrt{2}}{3\pi^{3/2}} \frac{a_2}{a_1^2} + O(\varepsilon^2) \ , \quad \text{if } a_2 \neq 0$

(4.8c) $\lambda = \lambda_1 - \varepsilon^2 \frac{3a_3}{2\pi a_1^2} + O(\varepsilon^3) \ , \quad \text{if } a_2 = 0 \text{ and } a_3 \neq 0 \ .$

We denote the bifurcation sub-branches that are approximated by (4.8) by $y = Y_1^{\pm}(x;\varepsilon)$. We observe that the bifurcation is

subcritical if $a_2 \neq 0$ and is supercritical (subcritical) if $a_2 = 0$ and $a_3 < 0$ (> 0).

The Perturbed Problem ($\delta \neq 0$).

The outer expansions of the perturbed problem are obtained by inserting

$$(4.9) \qquad y(x;\lambda,\delta) = \sum_{j=0}^{\infty} y_j(x;\lambda)\delta^j$$

into (4.1), where $y_0(x;\lambda)$ is any solution sub-branch of the bifurcation problem (4.3). This gives

$$(4.10a) \qquad F_y^o y_1 \equiv y_1'' + \lambda G'(y_0)y_1 = -\lambda g(x,y_0) \equiv R_1(x;\lambda)$$

$$y_1(0;\lambda) = y_1(\pi;\lambda) = 0 ,$$

$$(4.10b) \qquad F_y^o y_j = R_j(x;\lambda) , \quad y_j(0;\lambda) = y_j(\pi;\lambda) = 0 ,$$

$$j = 2,3,\ldots .$$

Here the inhomogeneous terms R_j are determined by $y_0, y_1, \ldots, y_{j-1}$.

We solve these linear, boundary value problems by using eigenfunction expansions. We employ the eigenfunctions v_m of the operator F_y^o. That is, the v_m are determined by

$$(4.11) \qquad v_m'' + \lambda G'(y_0)v_m - \sigma_m v_m = 0 , \quad v_m(0) = v_m(\pi) = 0 ,$$

where σ_m are the corresponding eigenvalues. Then by expanding the inhomogeneous terms R_j as

$$(4.12) \qquad R_j = \sum_{j=1}^{\infty} R_{j,m} v_m \ ,$$

we obtain the solutions of (4.10) as

$$(4.13) \qquad y_j(x;\lambda) = \sum_{m=1}^{\infty} \frac{R_{j,m}(\lambda)}{\sigma_m(\lambda)} v_m(x;\lambda) \ , \quad j = 1,2,\ldots \ .$$

For the branches $y_0 = Y^{\pm} \equiv 0$, we have $G'(y_0) = a_1$, and the eigenfunctions and eigenvalues of (4.11) are,

$$(4.14) \qquad v_m = \phi_m \ , \quad \sigma_m = \lambda a_1 - m^2 = a_1(\lambda - \lambda_m) \ ,$$

$$m = 1,2,\ldots \ .$$

We denote the outer expansions (4.9) corresponding to $Y_0^{\pm}$ by $y^{\pm}_{(0)}$. Then by using (4.13) and (4.14) we obtain,

$$(4.15) \qquad y^{\pm}_{(0)} = \left[-\frac{\lambda}{a_1} \sum_{m=1}^{\infty} \frac{g_m}{\lambda - \lambda_m} \phi_m \right] \delta + O(\delta^2) \ ,$$

where g_m is defined by

$$(4.16) \qquad g_m \equiv \int_0^{\pi} g(x,0)\phi_m(x)\,dx \ , \quad m = 1,2,\ldots \ .$$

In the notation of Section 2, g_1 is proportional to I, and hence by assumption (2.8), $g_1 \neq 0$. Thus as $\lambda \to \lambda_1$ from below, $y^{\pm}_{(0)}$ is positive (negative) if $g_1 > 0$ (< 0). The signs of $y^{\pm}_{(0)}$ are switched if $\lambda \to \lambda_1$ from above. The perturbed branches (4.15) are unbounded at the bifurcation point.

For the bifurcation branches $y_0 = Y_1^{\pm}(x;\varepsilon)$, $\lambda = \lambda(\varepsilon)$, given by (4.8), the coefficients of the differential equations

in (4.10a) and (4.11) depend on x and the parameter ε. Thus we solve the eigenvalue problem (4.11) by a perturbation expansion in ε. That is, we seek solutions of (4.11) in the form

$$(4.17)\qquad v_m(x;\varepsilon) = \sum_{j=0}^{\infty} v_{m,j}(x)\varepsilon^j\ , \quad \sigma_m(\varepsilon) = \sum_{j=0}^{\infty} \sigma_{m,j}\varepsilon^j\ ,$$

where $\langle v_m^2 \rangle = 1$. An analysis of the resulting system of linear boundary value problems for the $v_{m,j}$ yields the following:

$$(4.18)\qquad v_m = \phi_m + O(\varepsilon^2)\ ; \qquad m = 1,2,\ldots\ ;$$

for $a_2 \neq 0$,

$$(4.19a)\qquad \sigma_1 = \frac{4}{3}\left(\frac{2}{\pi}\right)^{3/2} \frac{a_2}{a_1}\varepsilon + O(\varepsilon^2)\ , \quad \sigma_m = (1-m^2) + O(\varepsilon)\ ,$$

$$m = 2,3,\ldots\ ;$$

for $a_2 = 0$, and $a_3 \neq 0$,

$$(4.19b)\qquad \sigma_1 = \frac{2a_3}{a_1}\varepsilon^2 + O(\varepsilon^3)\ , \quad \sigma_m = (1-m^2) + O(\varepsilon^3)\ ,$$

$$m = 2,3,\ldots\ .$$

To evaluate the coefficients in the outer expansions corresponding to the bifurcation branches from (4.13), we require the Fourier coefficients $R_{j,m}$. For example,

$$(4.20)\qquad R_{1,m} = \int_0^\pi R_1 v_m dx = -\lambda(\varepsilon)\int_0^\pi g[x,Y_1^{\pm}(x;\varepsilon)]v_m(x;\varepsilon)dx\ .$$

We insert the expansions (4.8), and (4.18) into (4.20), to obtain

$$(4.21) \qquad R_{1,m} = -a_1^{-1} g_m + O(\varepsilon) \; .$$

Finally, by substituting (4.8), (4.18), (4.19) and (4.21) into (4.13), and then substituting the result into (4.9), we obtain the outer expansions, which are denoted by $y^{\pm}_{(1)}$, as

$$y^{\pm}_{(1)} = \varepsilon\phi_1(x) + O(\varepsilon^2) - \left[\frac{3}{4}\left(\frac{\pi}{2}\right)^{3/2} \frac{g_1}{a_3\varepsilon}\,\phi_1 + O(1)\right]\delta$$

$$+ O(\delta^2) \; , \quad \text{if } a_2 \neq 0 \; ,$$

(4.23)

$$y^{\pm}_{(1)} = \varepsilon\phi_1(x) + O(\varepsilon^2) - \left[\left(\frac{\pi}{2}\right) \frac{g_1}{a_3\varepsilon^2}\,\phi + O(1/\varepsilon)\right]\delta$$

$$+ O(\delta^2) \; , \quad \text{if } a_2 = 0 \; ,$$

$$\text{and } a_3 \neq 0 \; .$$

We observe that the perturbed branches (4.23) are unbounded at the bifurcation point λ_1 ($\varepsilon = 0$).

The Inner Expansions.

The second functional derivative of the nonlinear operator (4.1) at the singular point $\lambda = \lambda_1$, $y = 0$ is $F^0_{zz} = 2\lambda_1 a_2$. Therefore, Q given by (2.23) is proportional to a_2, and the boundary value problem (4.1) has a quadratic nonlinearity if $a_2 \neq 0$. If $a_2 = 0$, and $a_3 \neq 0$, then it is easy to see that C given by (2.30) is not zero and the boundary value problem has a cubic nonlinearity at the bifurcation point.

First, we shall study (4.1) with $a_2 \neq 0$. Since $F^0_\lambda \equiv 0$, and hence $\Lambda = 0$, the analysis of Section 2 for $\delta > 0$, shows

that we must seek solutions of the perturbed problem in the form

(4.24) $$z(x;\mu) = y(x;\lambda(\mu),\delta(\mu)) = \sum_{j=1}^{\infty} z_j(x)\mu^j ,$$

$$\text{and} \quad \lambda = \lambda_1 + \xi\mu + \sum_{j=2}^{\infty} \xi_j\mu^j , \quad \delta = \mu^2 .$$

The coefficients z_j then satisfy,

(4.25) $$F_z^o z_1 \equiv z_1'' + z_1 = \rho_1 \equiv 0 , \qquad z_1(0) = z_1(\pi) = 0 ,$$

(4.26) $$F_z^o z_2 = \rho_2 \equiv -(a_2/a_1)z_1^2 - \xi a_1 z_1 - g(x,0)/a_1 ,$$

$$z_2(0) = z_2(\pi) = 0 ,$$

(4.27) $$F_z^o z_j = \rho_j , \qquad z_j(0) = z_j(\pi) = 0 .$$

Thus z_1, is given by

(4.28) $$z_1 = A\phi_1(x) ,$$

where A satisfies the quadratic equation

(4.29) $$A^2 + 2p\xi A + q = 0 ,$$

$$\text{where} \quad 2p \equiv \frac{3}{4}\left(\frac{\pi}{2}\right)^{3/2} \frac{a_1^2}{a_2} , \quad q \equiv \frac{3}{4}\left(\frac{\pi}{2}\right)^{3/2} \frac{g_1}{a_2} .$$

There are two real roots of (4.29) if,

(4.31) $$\xi^2 > (q/p^2) .$$

Thus there are two inner expansions

$$z^{\pm} = A^{\pm}(\xi)\sin x\ \delta^{1/2} + O(\delta^2) \ , \tag{4.32}$$
$$\lambda = \lambda_1 + \xi\delta^{1/2} + O(\delta) \ .$$

If $a_2 g_1 < 0$, then these expansions are defined for all ξ. If $a_2 g_1 > 0$, then there are two expansions for $\xi > q^{1/2}/p$, and two expansions for $\xi < -q^{1/2}/p$. At $|\xi| = q^{1/2}/p$, $|dA^{\pm}/d\xi| = \infty$.

The Matching.

To apply the matching condition, we first express the outer expansions in terms of the inner variables as in (2.35). Thus substituting (4.24) into (4.13), (4.15), and (4.16), and (4.23), and rearranging the result in a power series in μ, we get,

$$y^{\pm}_{(0)} = -\frac{g_1}{a_1^2\xi}\phi_1\mu + O(\mu^2)$$
$$y^{\pm}_{(1)} = -\left[2p\xi - \frac{g_1}{a_1^2}\xi^{-1}\right]\phi_1\mu + O(\mu^2) + \dots \ . \tag{4.33}$$

The asymptotic formsof the inner coefficients $z_1^{\pm}$ are,

$$\lim_{\xi\to\infty} z_1^{+} = \lim_{\xi\to-\infty} z_1^{-} = -\frac{q}{2p\xi}\phi_1(x) \ ,$$
$$\lim_{\xi\to\infty} z_1^{-} = \lim_{\xi\to-\infty} z_1^{+} = -2p\xi\phi_1(x) \ . \tag{4.34}$$

Since η_1 is the coefficient of μ in (4.33), it follows from (4.33) and (4.34) and the matching condition (2.37) with $k = 1$ that the branches connect as summarized in Table II. The resulting composite expansions for $g_1 > 0$ and $g_1 < 0$ are shown in Figures 6.

Table II

$\xi_1 \to$	∞	$-\infty$
z^+	$y^+_{(0)}$	$y^-_{(1)}$
z^-	$y^+_{(1)}$	$y^-_{(0)}$

The analysis of (4.1) with $a_2 = 0$ and $a_3 \neq 0$ (the cubic nonlinearity) is similar, and hence we present the results only. The stretching variables are now

(4.35) $$\lambda = \lambda_1 + \xi\mu^2 + O(\mu^4) , \qquad \delta = \mu^3 .$$

The inner expansions are, c.f. (2.27)

$$z = A\phi_1\mu + O(\mu^2) ,$$

where A is any real root of the cubic, c.f. (2.29b)

(4.36) $$A^3 + 2r\xi A + s = 0 ,$$

$$\text{where} \quad r \equiv \frac{\pi a_1^2}{3a_3} , \quad s \equiv \frac{2\pi}{3a_3} g_1 .$$

Thus if $a_3 < 0$ $(r < 0)$, which corresponds to supercritical bifurcation, we obtain essentially, the cubic (3.17). The graphs of the resulting composite expansions are similar to Figure 4c. If $a_3 > 0$ $(r > 0)$, so that we have subcritical bifurcation, the analysis is equivalent essentially to that of (3.22). Thus the graphs of the resulting composite expansions are qualitatively similar to Figure 5c.

Interpretation.

To interpret the solutions of the perturbed problem, we consider (4.1) as describing the equilibrium states of a diffusion process, so that we study only non-negative solutions. The stability of these equilibrium states is determined from the linearized, time dependent, diffusion problem that is linearized about a solution $y(x)$ of (4.1).

First we consider (4.1) with $a_2 > 0$. The composite solutions are shown in Figures 6. For the bifurcation problem $(\delta = 0)$, the zero-concentration solution, $y \equiv 0$, is stable for $\lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$. Thus for the bifurcation problem, as λ (the source strength) exceeds λ_1, the zero-concentration state is unstable, and a transient diffusion occurs. For the perturbed problem $(\delta \neq 0)$ with $g_1 > 0$, Figure 6a, as λ increases from zero there is a "small-concentration" equilibrium state. When λ exceeds λ_u,

there is no near-by equilibrium state and transient diffusion must occur. Thus λ_1 and λ_u are the limits of the source strengths for the small-concentration equilibrium states of the bifurcation and perturbed problems. Since $\lambda_u - \lambda_1 = O(\delta^{1/2})$, we see that small perturbations, $O(\delta)$ in the model (4.1), lead to relatively large, $O(\delta^{1/2})$, changes in the response. Thus diffusion models with $a_2 > 0$ are sensitive to small perturbations with $g_1 > 0$. A similar phenomenon occurs in the buckling of elastic plates and shells. Structures that exhibit such a sensitivity are called imperfection-sensitive structures.

The diffusion model is even more sensitive for perturbations with $g_1 < 0$, see Figure 6b. For then there are no small concentration equilibrium states as λ increases from zero. Thus transient diffusion occurs immediately as λ increases.

For diffusion models with $a_2 = 0$, $a_3 \neq 0$, and that have solution branches as sketched in Figure 4c, small perturbations in the model give relatively small changes in the small-concentration equilibrium states. The perturbed branch gives a smooth transition from the zero-concentration to the non-zero-concentration equilibrium states of the bifurcation problem. Such models are said to be insensitive to small perturbations. Furthermore, transient diffusion does not occur as λ increases, since the equilibrium states are stable.

Appendix I. A Model of Elastic Arch Buckling.

Two rigid and massless rods of length ℓ are connected by a linear torsional spring, as shown in the Figure 7. The total mass, m, of the system is concentrated at the junction point. The left and right endpoints of the rods are pinned, so that they cannot move vertically, but they are free to slide horizontally. A compressive force T is applied to the ends of the rods, as shown in the figure. The initial position of the rods, before application of the force is shown by the dotted lines. Thus in the initial position, the junction point is a specified distance w_o above the horizontal and the rods have an initial slope θ_o. As a result of the force the junction point is a distance w above the horizontal and the deformed slope is θ.

The motion of the rods is resisted by the torsional spring, which has a spring constant Γ, and a nonlinear extensional spring. Then the torsional resistance moment M, is proportional to the change in angle of intersection of the rods. Thus we have,

$$M = \Gamma(\theta-\theta_o) .$$

The force in the extensional spring is a function of the relative displacement $w - w_o$. In particular, we shall consider the nonlinear law

$$F = a(w-w_o) + b(w-w_o)^2 + c(w-w_o)^3 ,$$

where a, b and c are the prescribed spring constants. If $b \neq 0$, then the spring is unsymmetrical.

Since $\sin\theta = w/\ell$ and $\sin\theta_o = w_o/\ell$, we have for small θ and θ_o,

$$\theta \simeq w/\ell \ , \qquad \theta_o \simeq w_o/\ell \ .$$

Then by defining a dimensionless variable y by

$$y \equiv \frac{w-w_o}{A} \ , \qquad A \equiv [(a+4\Gamma/\ell^2)/c]^{1/2} \ ,$$

the equation of motion of the mass is,

$$\text{(A.1)} \qquad \ddot{y} + y^3 + 2\beta y^2 + (1-\lambda)y - \lambda\delta = 0 \ .$$

The variable y is a dimensionless "excess" deflection of the mass. The constants β, λ and δ in (4.1) are defined by

$$\text{(A.2)} \qquad \beta \equiv \frac{b}{2Ac} \ , \qquad \lambda \equiv \frac{2T}{cA^2} \ , \qquad \delta \equiv \frac{w_o}{A} \ .$$

The parameter λ, which is proportional to the thrust, is called the load parameter, and δ is the imperfection parameter. For equilibrium states $\ddot{y} = 0$, and Equation (A.1) is reduced to (3.1).

References

1. K. O. Friedrichs, Asymptotic Phenomena in Mathematical Physics, Bul. AMS, 61 (1955), pp. 485-504.

2. J. Cole, "Perturbation Methods in Applied Mathematics", Blaisdell, New York, 1968.

3. R. E. O'Malley, "Introduction to Singular Perturbations", Academic Press, 1975.

4. W. T. Koiter, "On the Stability of Elastic Equilibrium" (in Dutch), Delft, 1945; translated into English as NASA TTF-10, 833 (1967).

5. J. P. Keener and H. B. Keller, Perturbed Bifurcation Theory, Arch. Rational Mech. Anal., 50 (1973), pp. 159-175.

6. J. B. Keller, Nonlinear Forced and Free Vibrations in Acoustic Waveguides, J. Acoust. Soc. Am., 55 (1974), pp. 524-527.

7. S. N. Chow, J. K. Hale and J. Mallet-Paret, Applications of Generic Bifurcation, I and II, Arch. for Rational Mech. Anal., 59 (1975), pp. 159-188; 62 (1976), pp. 209-235.

8. R. Thom, Topological Methods in Biology, Topology, 8 (1968), pp. 313-335.

9. M. M. Vainberg and V. A. Trenogin, "Theory of Branching of Solutions of Nonlinear Equations", Translation, Noordhoff International Pub., Leyden, Netherlands, 1974.

10. Reiss, E. L., Imperfect Bifurcation, in Advanced Seminar on Bifurcation Theory, Univ. of Wisc., to appear in 1977.

11. Bauer, L., Keller, H. B. and Reiss, E. L., Multiple Eigenvalues Lead to Secondary Bifurcation, SIAM Rev., 17 (1975), pp. 101-122.

12. Mahar, T. and Matkowsky, B. J., A Model Biochemical Reaction Exhibiting Secondary Bifurcation, SIAM J. Appl. Math., 1977.

Captions for Figures

Figure 1. None

Figure 2a. The inner and outer solutions for the perturbed Equation (3.1), with $\delta > 0$. The inner branches are represented by the dashed curves, and the outer branches by the heavy solid curves. The light solid curves represent the bifurcation branches. In these figures, we have taken β in the interval, $0 < \beta < 1$.

Figure 2b. The bifurcation branches ($\delta = 0$) for the elastic arch model, Equation (3.1). $Y_0^{\pm}$ are the sub-branches of Y_0, and Y_1^{++} and Y_1^{+-} are the sub-branches of Y_1.

Figure 2c. The inner and outer branches for Equation (3.1) with $\delta < 0$.

Figure 3a. The composite solutions for Equation (3.1) with $\delta > 0$.

Figure 3b. The composite solutions for Equation (3.1) with $\delta < 0$ and $\xi^o = 2\sqrt{2\beta}$.

Figure 4a. The bifurcation branches ($\delta = 0$) for Equation (3.12).

Figure 4b. The inner and outer solutions for Equation (3.12) with $\delta > 0$.

Figure 4c. The composite solutions for Equation (3.12) with $\delta > 0$. The graphs of the composite solutions for Equation (3.12) with $\delta < 0$ are the images with respect to the λ axis of the curves shown in the figure.

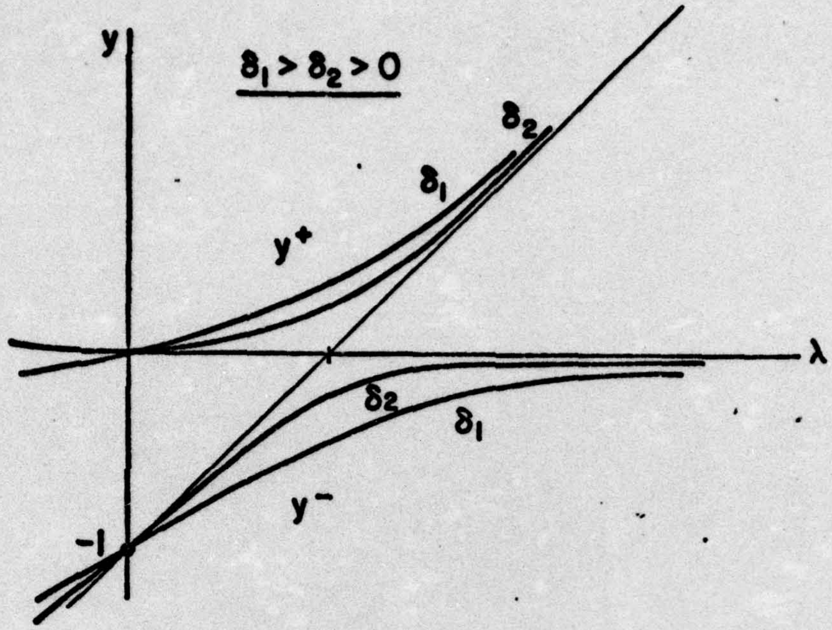
Figure 5a. The bifurcation branches ($\delta = 0$) for Equation (3.22).

Figure 5b. The inner and other solutions for Equation (3.22) with $\delta > 0$.

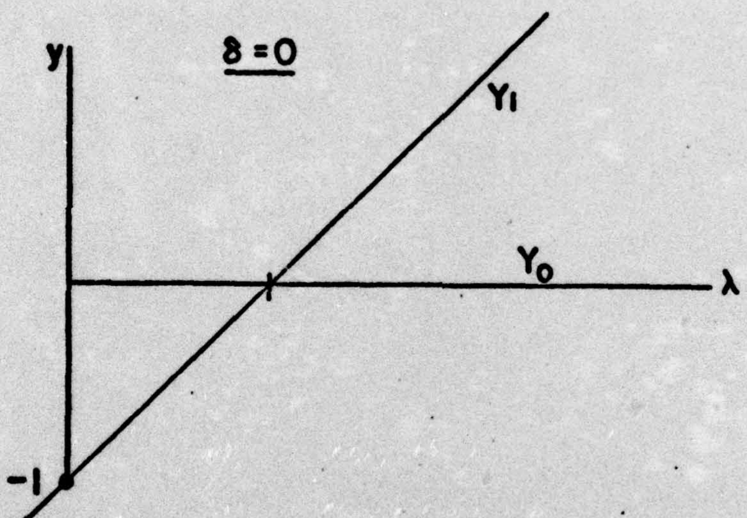
Figure 5c. The composite solutions for Equation (3.22) with $\delta > 0$. The images of these curves with respect to the λ axis are the graphs of the composite solutions for (3.22) with $\delta < 0$.

Figure 6. The composite solutions for the nonlinear boundary value problem (4.1) with $a_2 > 0$. The noses of the branches for $q_1 > 0$ are $\lambda_u = \lambda_1 - (q^{1/2}/p)\delta^{1/2} + \cdots$, and $\lambda_L = \lambda_1 + (q^{1/2}/p)\delta^{1/2} + \cdots$.

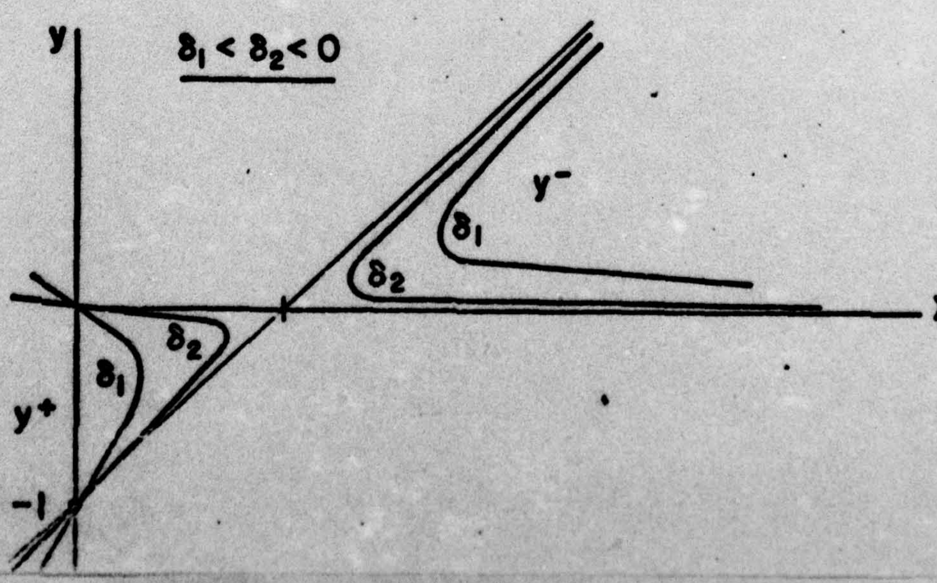
Figure 7. Model of an elastic arch.


y
$\delta_1 > \delta_2 > 0$
δ_2
δ_1
y^+
λ
δ_2
δ_1
y^-
-1

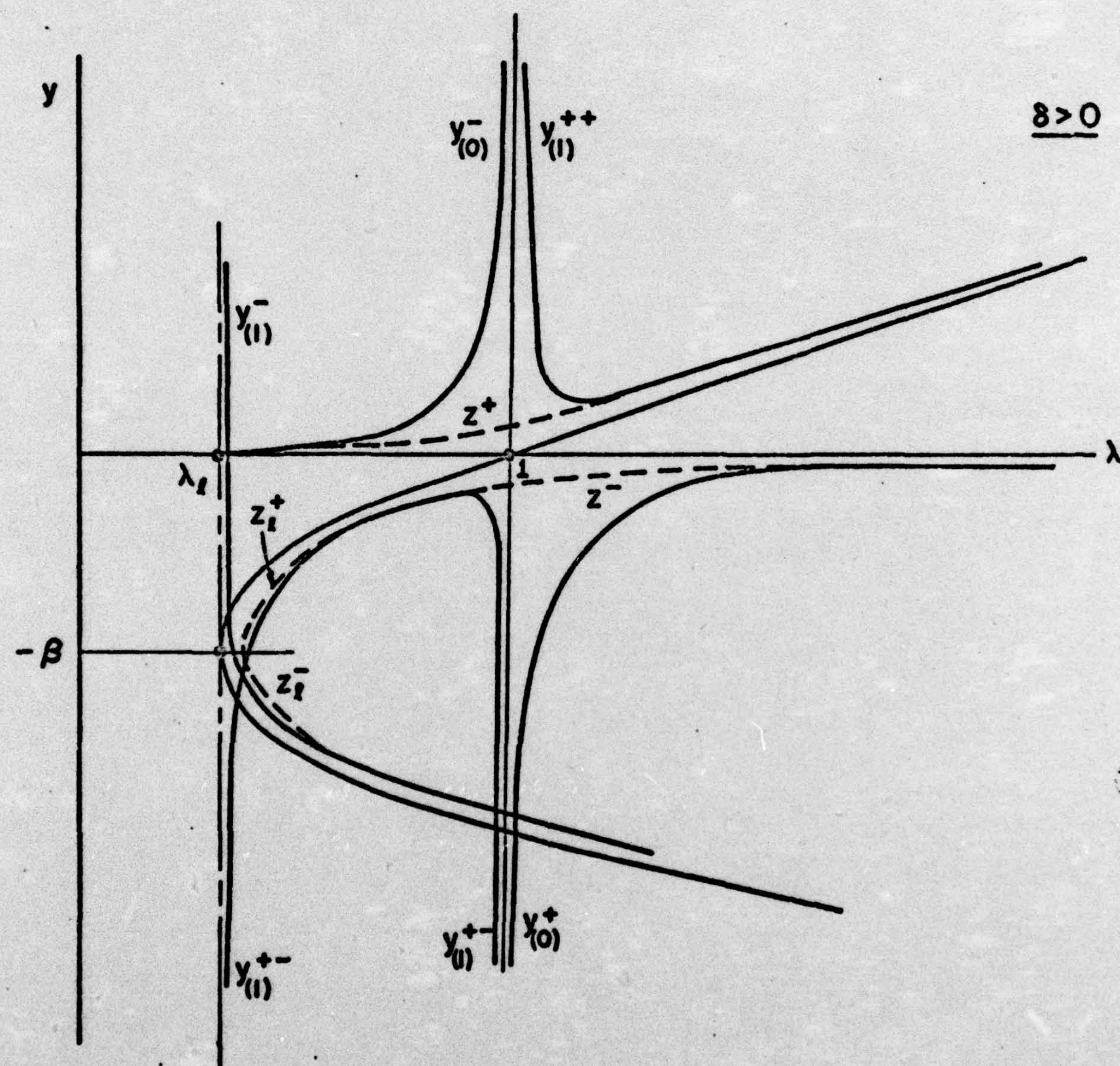
(1a)


y
$\delta = 0$
Y_1
Y_0
λ
-1

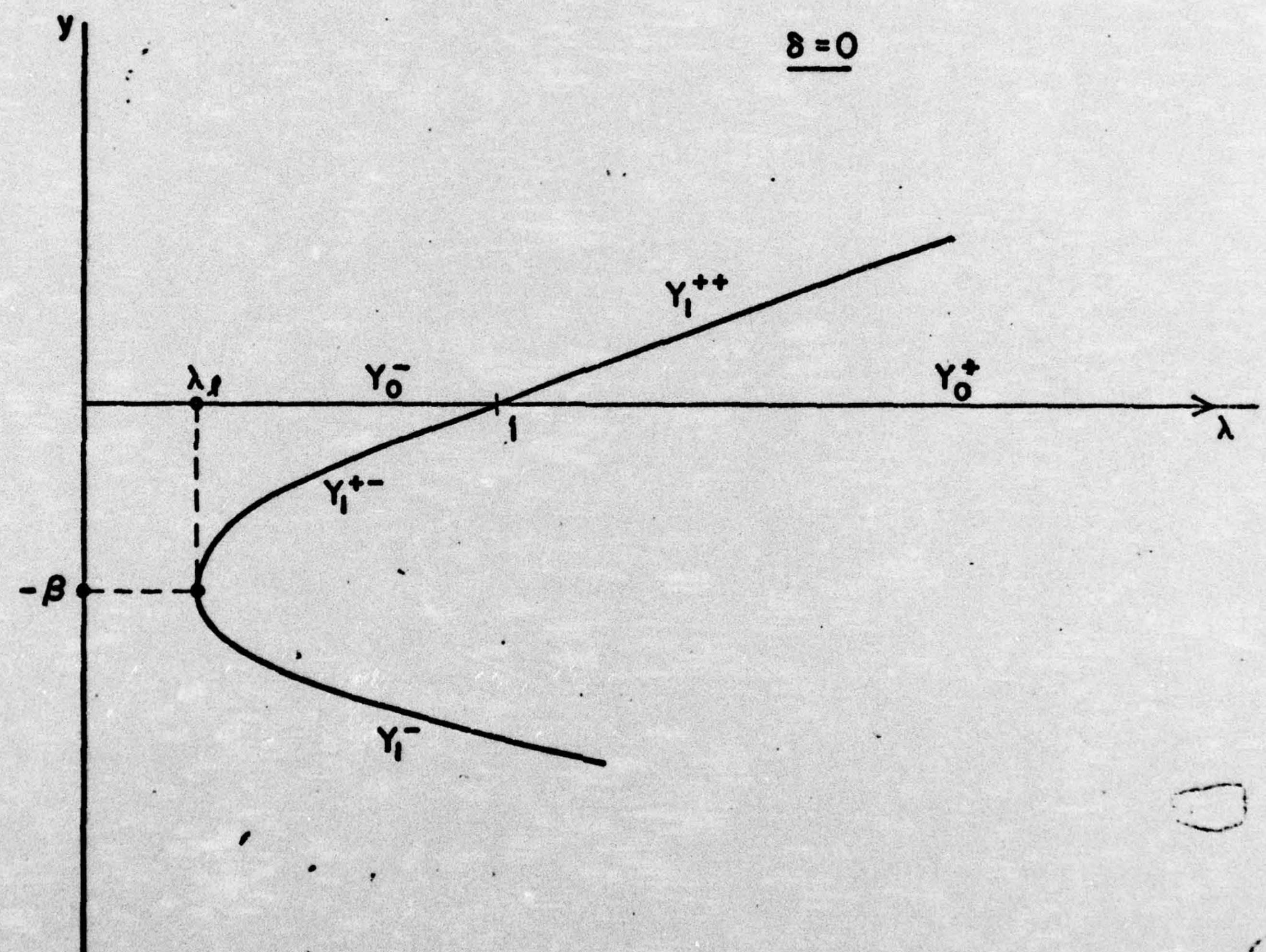
(1b)


y
$\delta_1 < \delta_2 < 0$
y^-
δ_1
δ_2
λ
δ_2
δ_1
y^+
-1

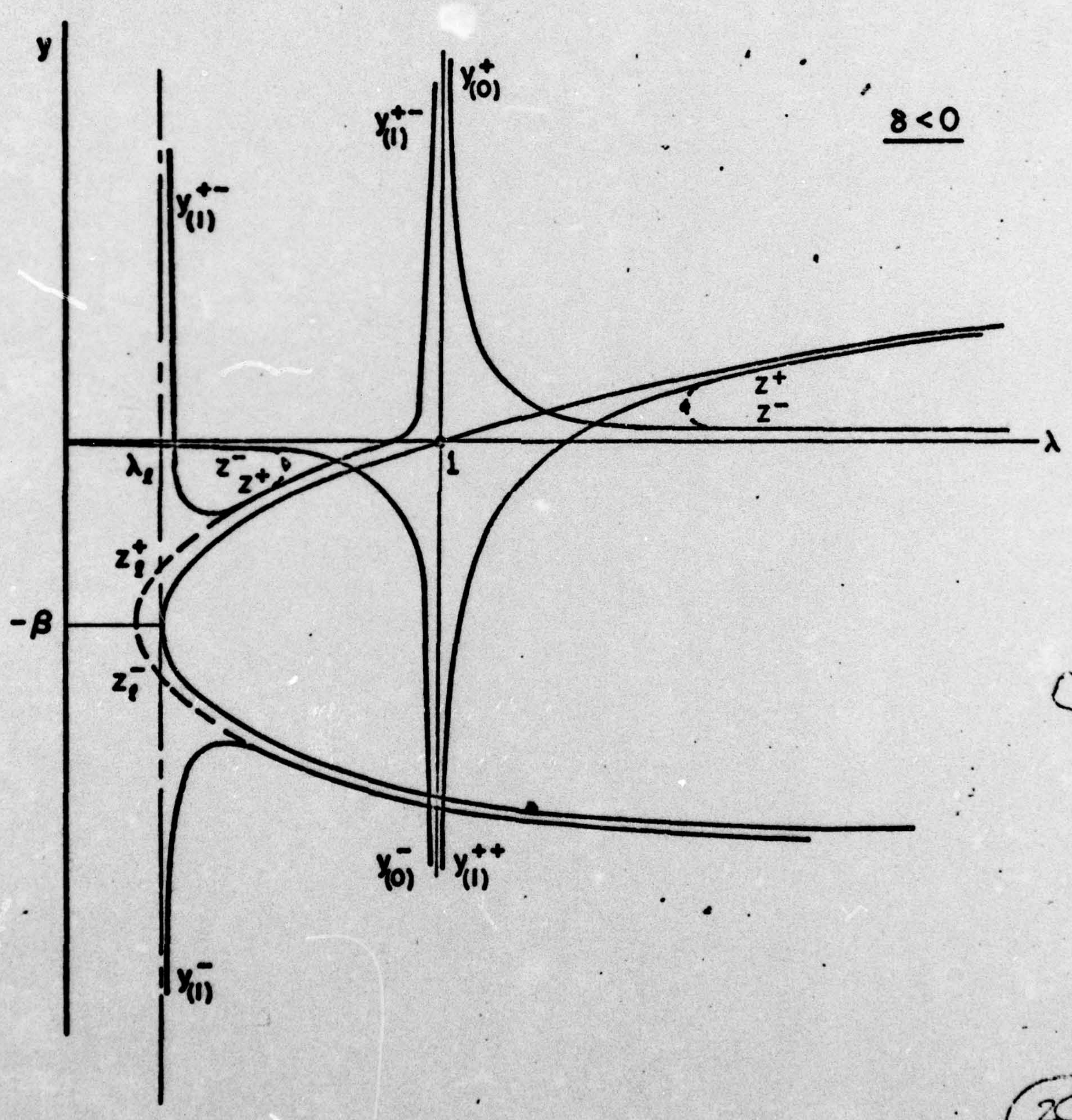
(1c)



y
$\delta > 0$
$y^-_{(0)}$
$y^{++}_{(1)}$
$y^-_{(1)}$
z^+
λ
λ_ℓ
1
z^-
z^+_ℓ
$-\beta$
z^-_ℓ
$y^{+-}_{(1)}$
$y^+_{(0)}$
$y^{+-}_{(1)}$

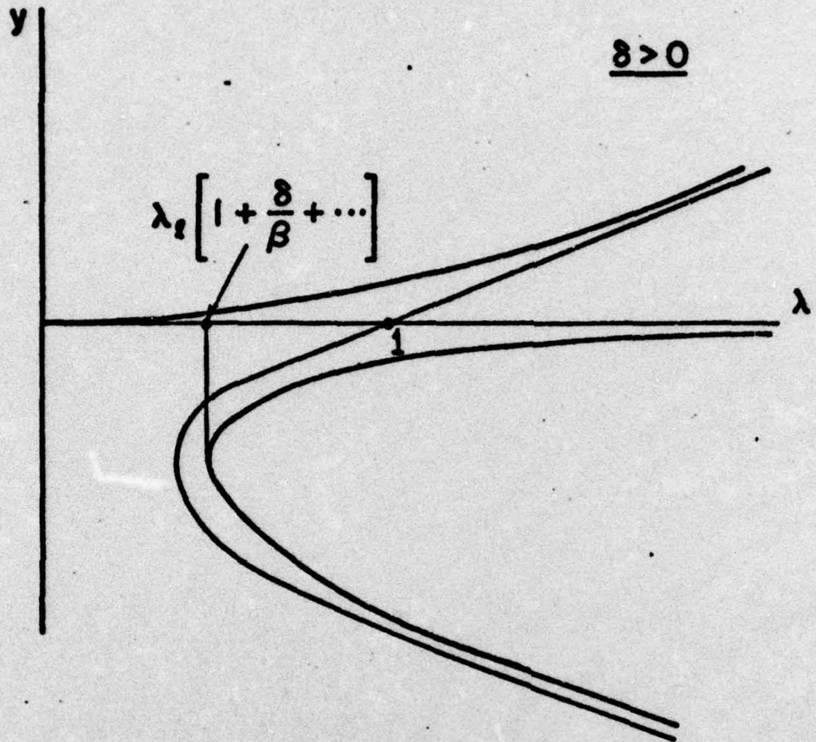


y
δ = 0
Y_1^{++}
λ_ℓ
Y_0^-
Y_0^+
1
λ
Y_1^{+-}
$-\beta$
Y_1^-

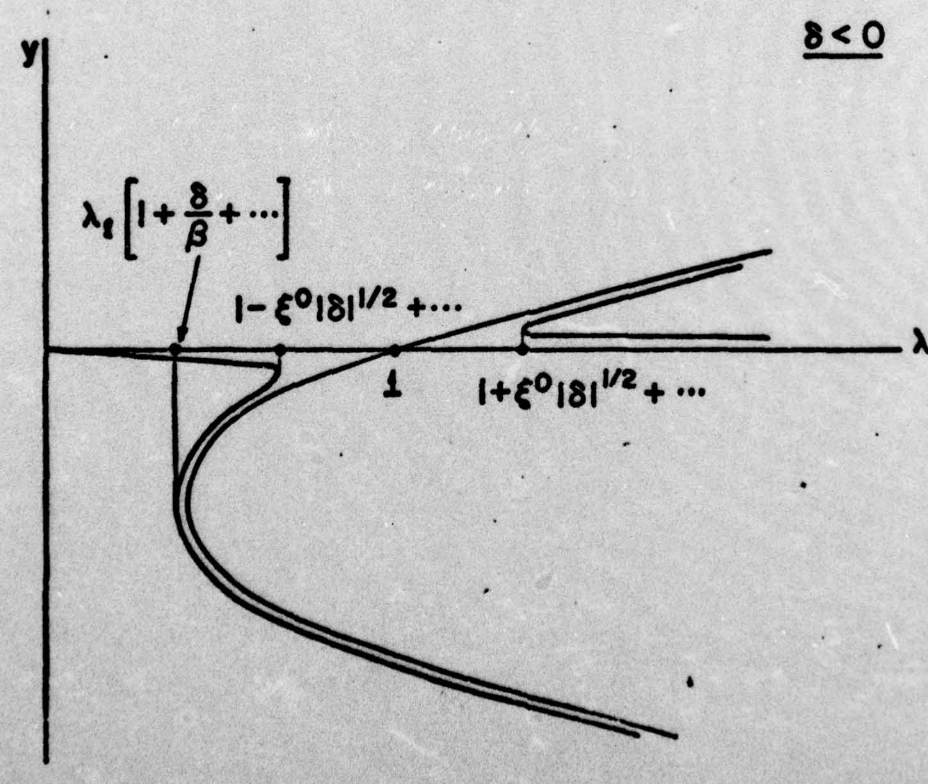


$\delta < 0$
y
λ
$y^{+}_{(0)}$
$y^{+-}_{(1)}$
$y^{+-}_{(1)}$
z^{+}
z^{-}
λ_{ℓ}
z^{-}
z^{+}
1
z^{+}_{ℓ}
$-\beta$
z^{-}_{ℓ}
$y^{-}_{(0)}$
$y^{++}_{(1)}$
$y^{-}_{(1)}$

2C



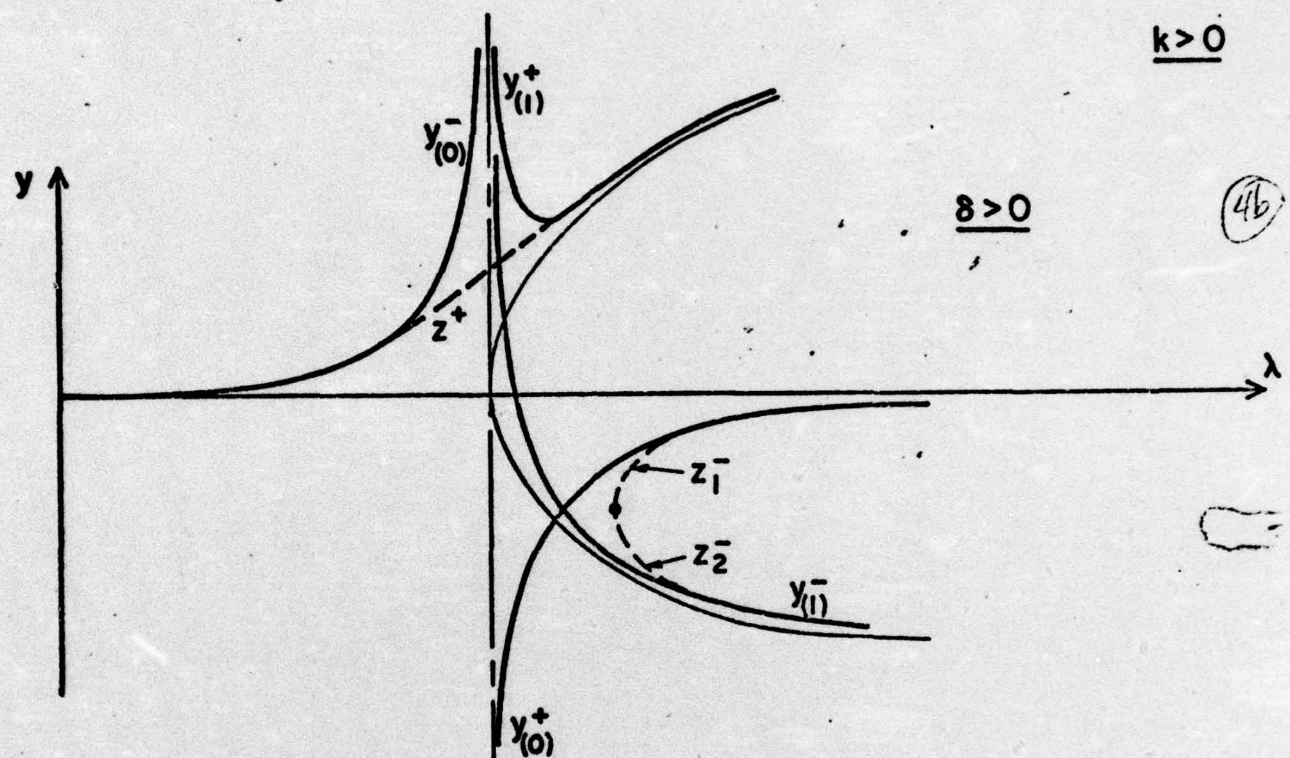
y
δ>0
λ
$\lambda_\ell\left[1+\frac{\delta}{\beta}+\cdots\right]$
1



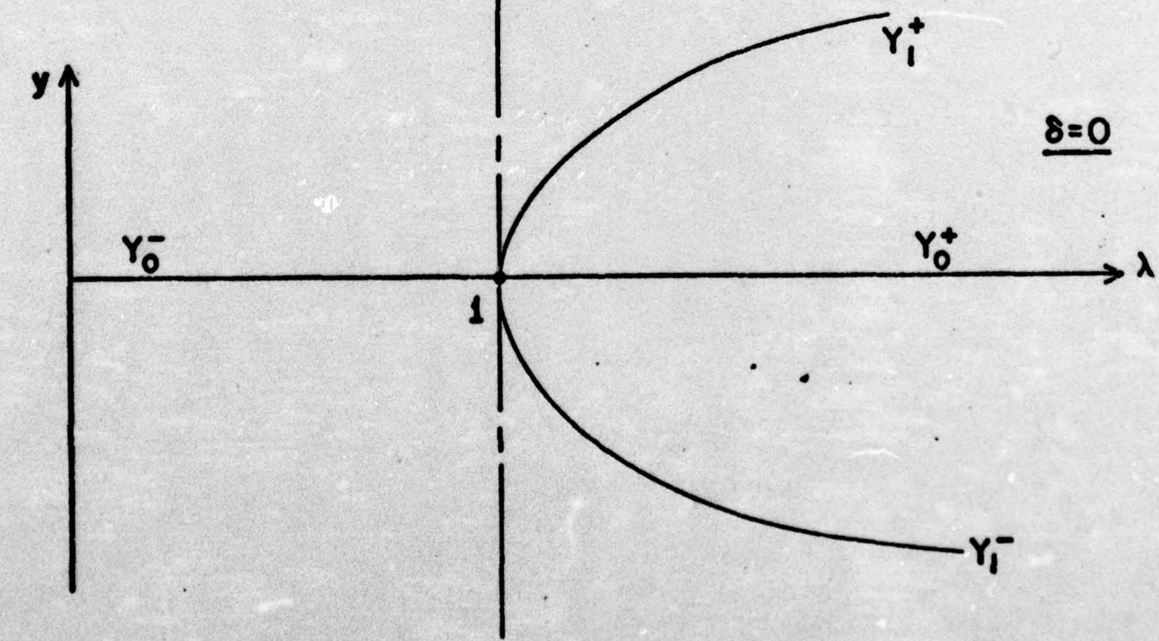
y
δ<0
λ
$\lambda_\ell\left[1+\frac{\delta}{\beta}+\cdots\right]$
$1-\xi^0|\delta|^{1/2}+\cdots$
1
$1+\xi^0|\delta|^{1/2}+\cdots$

30

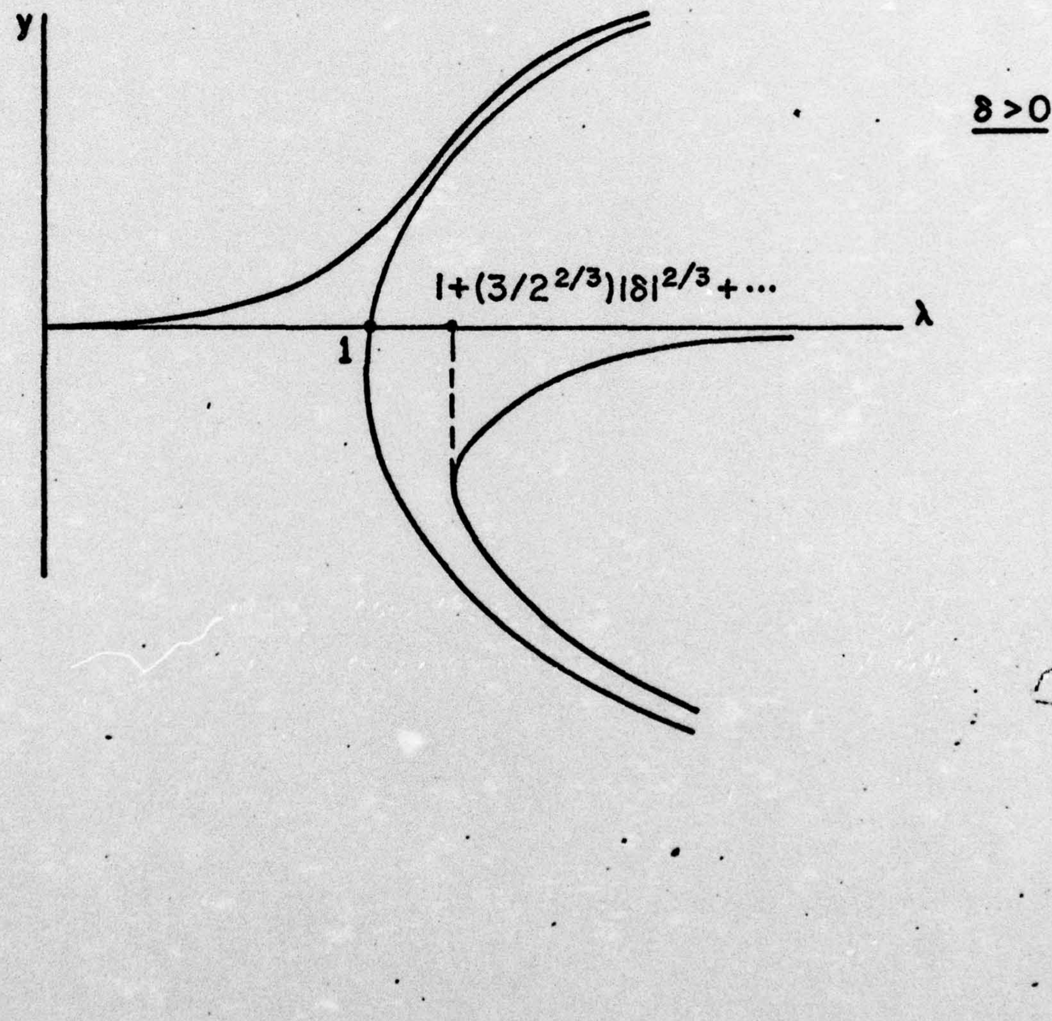
k > 0



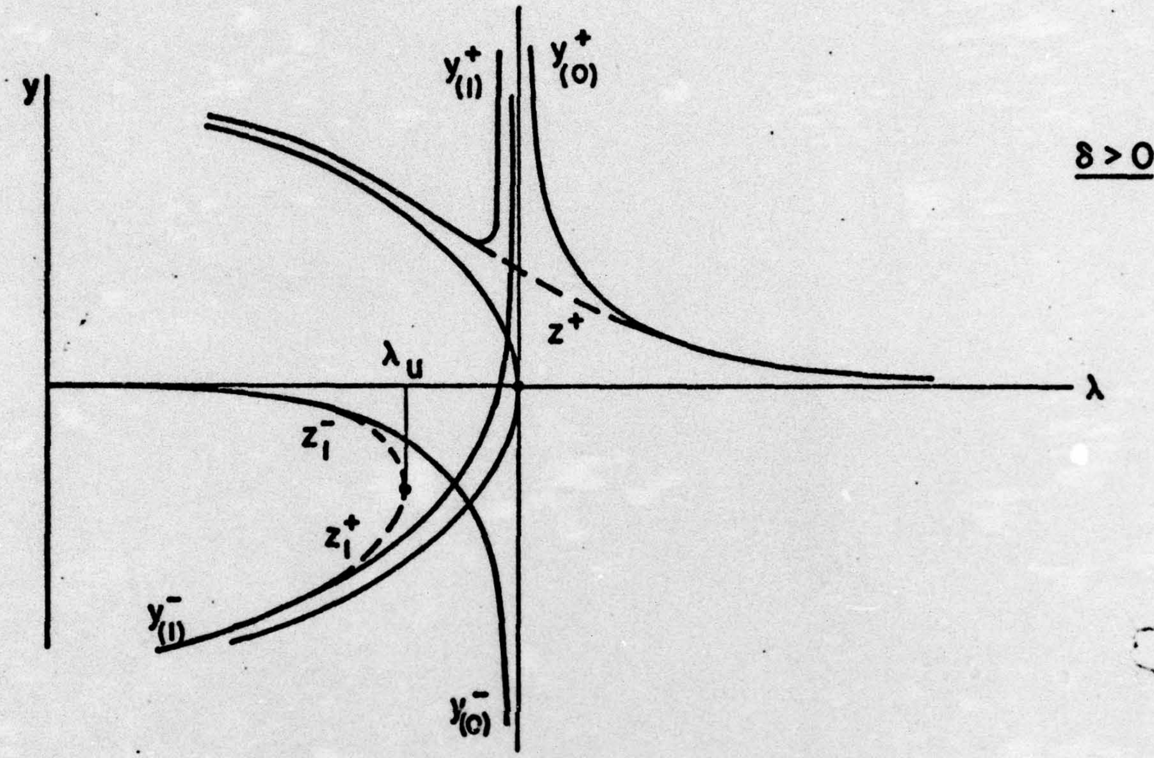
(4b)



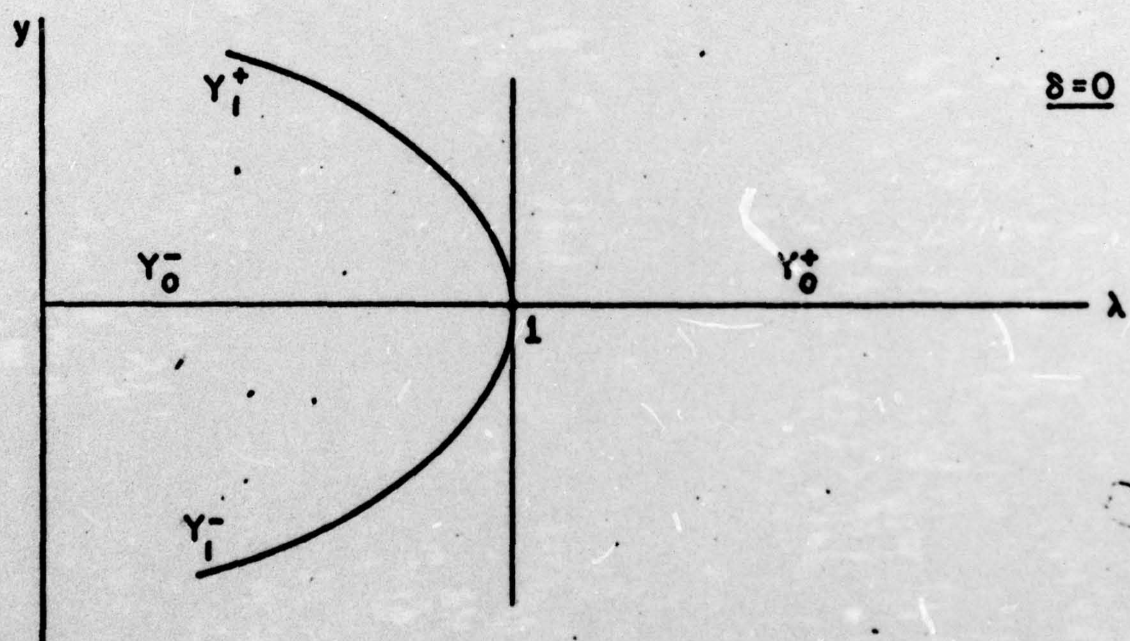
(4c)



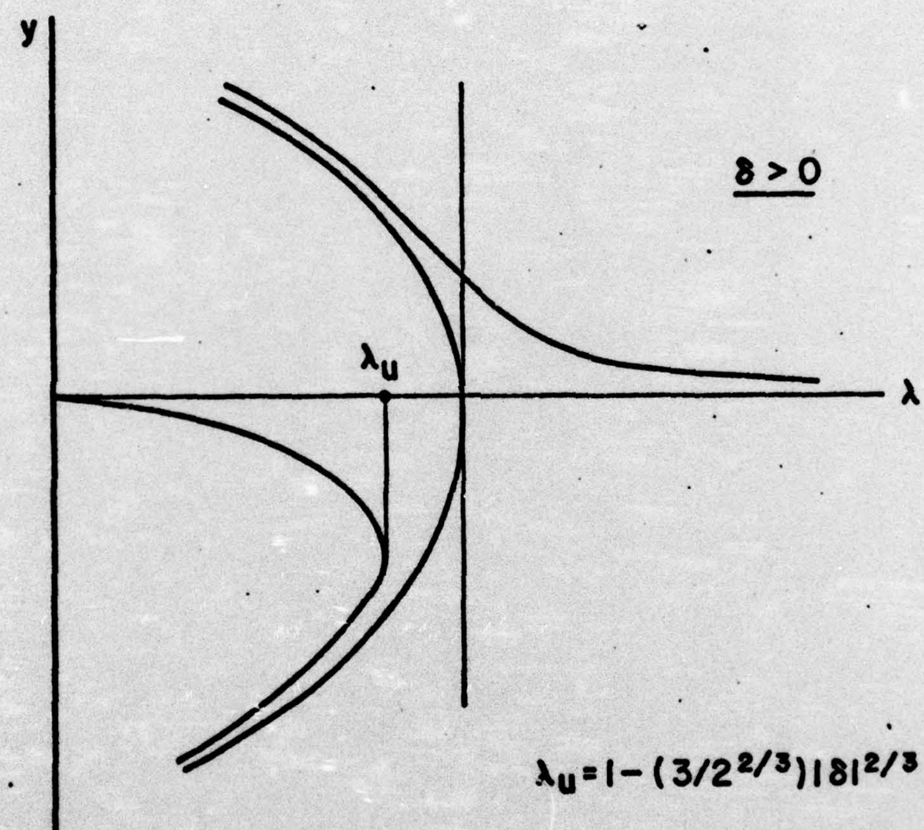
(4C)



y
$y^+_{(1)}$
$y^+_{(0)}$
$\delta > 0$
z^+
λ_u
λ
z^-_1
z^+_1
$y^-_{(1)}$
$y^-_{(0)}$

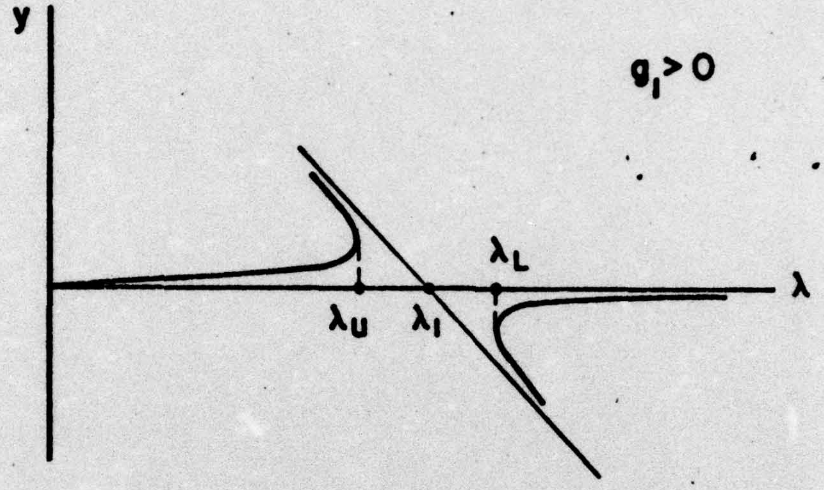


y
Y^+_1
$\delta = 0$
Y^-_0
Y^+_0
λ
1
Y^-_1

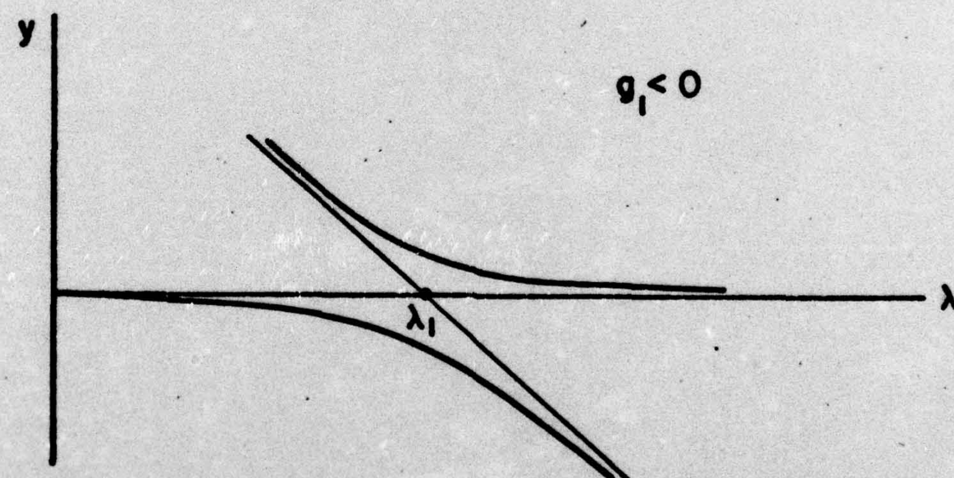


$\underline{\delta > 0}$

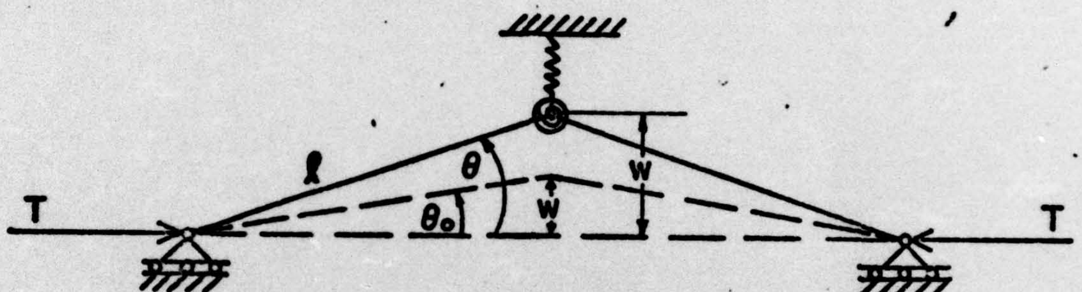
$\lambda_u = 1 - (3/2^{2/3})|\delta|^{2/3} + \cdots$



6a



6b



7